3 Network Analysis

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(Sections 3.3.1–3.3.5)

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3.1 Introduction

In an electrical network, electrical energy is conveyed from *sources* to an array of interconnected *branches* in which energy is converted, dissipated or stored. Each branch has a characteristic voltage–current relation that defines its *parameters*. The analysis of networks is concerned with the solution of source and branch currents and voltages in a given *network configuration*. Basic and general network concepts are discussed in Section 3.2. Section 3.3 is concerned with the special techniques applied in the analysis of power-system networks.

3.2 Basic network analysis

3.2.1 Network elements

Given the sources (generators, batteries, thermocouples, etc.), the network configuration and its branch parameters, then the network solution proceeds through network equations set up in accordance with the Kirchhoff laws.

3.2.1.1 Sources

In most cases a source can be represented as in *Figure 3.1*(*a*) by an electromotive force (e.m.f.) E_0 acting through an internal series impedance Z_0 and supplying an external 'load' Z with a current I at a terminal voltage V. This is the Helmholtz–Thevenin *equivalent voltage generator*. As regards the load voltage V and current I, the source could equally well be represented by the Helmholtz–Norton *equivalent current generator* in *Figure 3.1*(*b*), comprising a source current I_0 shunted by an internal admittance Y_0 which is effectively in parallel with the load of admittance Y. Comparing the two forms for the same load current I and terminal voltage V in a load of impedance Z or admittance $Y = 1/Z$, we have:

Voltage generator	*Current generator*
$V = E_0 - IZ_0$	$I = I_0 - VY_0$
$I = (E_0 - V)/Z_0$	$V = (I_0 - I)/Y_0$
$= E_0/Z_0 - V/Z_0$	$= I_0/Y_0 - I/Y_0$
$= I_0 - VY_0$	$= E_0 - IZ_0$

These are identical provided that $I_0 = E_0/Z_0$ and $Y_0 = 1/Z_0$. The identity applies only to the *load* terminals, for internally the sources have quite different operating conditions. The two forms are *duals*. Sources with $Z_0 = 0$ and $Y_0 = 0$ (so that $V = E_0$ and $I = I_0$) are termed *ideal* generators.

3.2.1.2 Parameters

When a real *physical* network is set up by interconnecting sources and loads by conducting wires and cables, all parts (including the connections) have associated electric and magnetic fields. A resistor, for example, has resistance as the prime property, but the passage of a current implies a magnetic field, while the potential difference (p.d.) across the resistor implies an electric field, both fields being present in and around the resistor. In the *equivalent* circuit drawn to represent the physical one it is usual to lump together the significant resistances into a limited number of *lumped* resistances. Similarly, electric-field effects are represented by lumped capacitance and magnetic-field effects by lumped inductance. The equivalent circuit then behaves like the physical prototype if it is so constructed as to include all significant effects.

The lumped parameters can now be considered to be free from 'residuals' and *pure* in the sense that simple laws of behaviour apply. These are indicated in *Figure 3.2*.

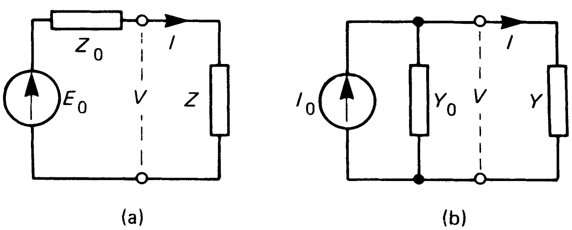


Figure 3.1 (a) Voltage and (b) current sources

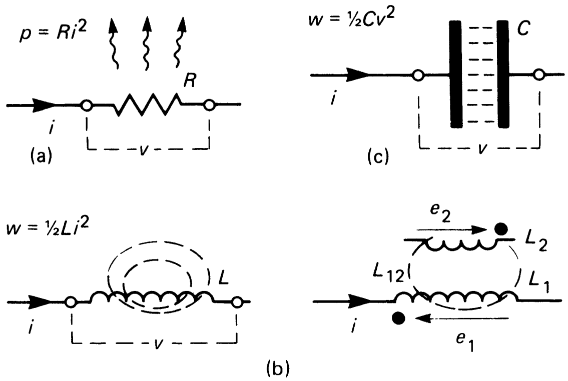


Figure 3.2 Pure parameters

(*a*) *Resistance* For a pure resistance R carrying an instantaneous current i, the p.d. is $v = Ri$ and the rate of heat production is $p = vi = Ri^2$. Alternatively, if the conductance $G = 1/R$ is used, then $i = Gv$ and $p = vi = Gv^2$. There is a constant relation

$$v = Ri = v/G; \quad i = Gv = v/R; \quad p = Ri^2 = Gv^2$$

(*b*) *Inductance* With a *self*-inductance L, the magnetic linkage is Li, and the source voltage is required only when the linkage changes, i.e. $v = \mathrm{d}(Li)/\mathrm{d}t = L(\mathrm{d}i/\mathrm{d}t)$. An inductor stores in its magnetic field the energy $w = \frac{1}{2}Li^2$. The behaviour equations are

$$v = L(\mathrm{d}i/\mathrm{d}t); \quad i = (1/L)\int v\,\mathrm{d}t; \quad w = \tfrac{1}{2}Li^2$$

Two inductances L_1 and L_2 with a common magnetic field have a *mutual* inductance $L_{12} = L_{21}$ such that an e.m.f. is induced in one when current changes in the other:

$$e_1 = L_{12}(\mathrm{d}i_2/\mathrm{d}t); \quad e_2 = L_{21}(\mathrm{d}i_1/\mathrm{d}t) \Leftarrow$$

The direction of the e.m.f.s depends on the change (increase or decrease) of current and on the 'sense' in which the inductors are wound. The 'dot convention' for establishing the sense is to place a dot at one end of the symbol for L_1, and a dot at that end of L_2 which has the same polarity as the dotted end of L_1 for a given change in the common flux.

(*c*) *Capacitance* The stored charge q is proportional to the p.d. such that $q = Cv$. When v is changed, a charge must enter or leave at the rate $i = \mathrm{d}q/\mathrm{d}t = C(\mathrm{d}v/\mathrm{d}t)$. The electric-field energy in a charged capacitor is $w = \frac{1}{2}Cv^2$. Thus

$$i = C(\mathrm{d}v/\mathrm{d}t); \quad v = (1/C)\int i\,\mathrm{d}t; \quad w = \tfrac{1}{2}Cv^2$$

It can be seen that there is a duality between the inductor and the capacitor. Some typical cases of the behaviour of pure parameters are given in *Figures 2.3*, *2.21* and *2.28*.

A more concise representation of the behaviour of pure parameters uses the differential operator p for d/dt and the inverse 1/p for the integral operator: then

(*a*) Resistance: $v = Ri = v/G$; $i = Gv = (1/R)v$
(*b*) Self-inductance: $v = L\mathrm{p}i$; $i = (1/L\mathrm{p})v$
Mutual inductance: $e_1 = L_{12}\mathrm{p}i_2$; $e_2 = L_{21}\mathrm{p}i_1$
(*c*) Capacitance: $v = (1/C\mathrm{p})i$; $i = C\mathrm{p}v$

For the steady-state direct-current (d.c.) case, p = 0. For steady-state sinusoidal alternating current (a.c.), p = jω, giving for L and C the forms jωL and 1/jωC where ω is the angular frequency. In general, Lp and 1/Cp are the *operational* impedance parameters.

3.2.1.3 Configuration

The assembly of sources and loads forms a network of branches that interconnect nodes (junctions) and form meshes. The seven-branch network shown in *Figure 3.3* has five nodes (a, b, c, d, e) and four meshes (1, 2, 3, 4). Branch ab contains a voltage source; the other branches have (unspecified) impedance parameters. Inspection shows that not all the meshes are independent: mesh 4, for example, contains branches already accounted for by meshes 1, 2 and 3. Further, if one node (say, e) is taken as a *reference* node, the voltages of nodes a, b, c and d can be taken as their p.d.s with respect to node e. The network is then taken as having $b = 7$ branches, $m = 3$ independent meshes and $n = 4$ independent nodes. In general, $m = b - n$.

3.2.2 Network laws

The behaviour of networks (i.e. the branch currents and node voltages for given source conditions) is based on the two Kirchhoff laws (*Figure 3.4*).

(1) *Node law* The total current flowing into a node is zero, $\Sigma i = 0$. The sum of the branch currents flowing into a node must equal the sum of the currents flowing from it; this is a result of the 'particle' nature of conduction current.
(2) *Mesh law* The sum of the voltages around a closed mesh is zero, $\Sigma v = 0$. A rise of potential in sources is absorbed by a fall in potential in the successive branches forming the mesh. This is the result of the nature of a network as an energy system.

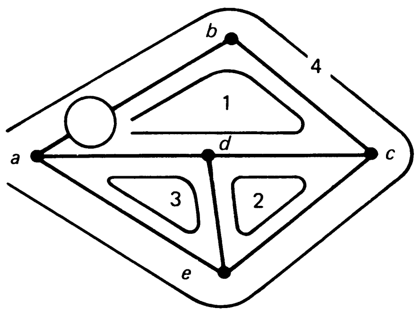


Figure 3.3 Network configuration

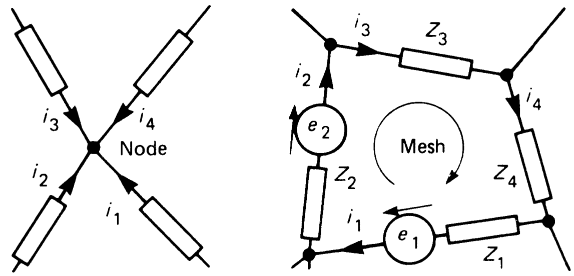


Figure 3.4 The Kirchhoff laws

The Kirchhoff laws apply to all networks. Whether the evaluation of node voltages and mesh currents is tractable or not depends not only on the complexity of the network configuration but also on the branch parameters. These may be active or passive (i.e. containing or not containing sources), linear or non-linear. Non-linearity, in which the parameters are not constant but depend on the voltage and/or current magnitude and polarity, is in fact the normal condition, but where possible the minor non-linearities are ignored in order to permit the use of greatly simplified analysis and the principle of superposition.

3.2.2.1 Superposition

In a strictly linear network, the current in any branch is the sum of the currents due to each source acting *separately*, all other sources being replaced meantime by their internal impedances. The principle applies to voltages and currents, but not to powers, which are current–voltage products.

3.2.3 Network solution

A general solution presents the voltages and currents everywhere in the network; it is initiated by the solution simultaneously of the network equations in terms of voltages, currents and parameters.

The Kirchhoff laws can be applied systematically by use of the *Maxwell circulating-current* process. To each mesh is assigned a circulating current, and the laws are applied with due regard to the fact that certain branches, being common to two adjacent meshes, have net currents given by the superposition of the individual mesh currents postulated. Generalising, the network can be considered as either (i) a set of independent nodes with appropriate node-voltage equations, or (ii) a set of independent meshes with corresponding mesh-current equations.

3.2.3.1 Mesh-current equations

This is a formulation of the Maxwell circulating-current process. If source e.m.f.s are written as E, currents as I and impedances as Z, then for the m independent meshes

$$E_1 = I_1 Z_{11} + I_2 Z_{12} + \cdots + I_m Z_{1m}$$
$$E_2 = I_1 Z_{21} + I_2 Z_{22} + \cdots + I_m Z_{2m}$$
$$\vdots \qquad \vdots \qquad \vdots$$
$$E_m = I_1 Z_{m1} + I_2 Z_{m2} + \cdots + I_m Z_{mm}$$

Here $Z_{11}, Z_{22}, \ldots, Z_{mm}$ are the *self-impedances* of meshes 1, 2, ..., m, i.e. the total series impedance around each of the chosen meshes; and $Z_{12}, Z_{pq}, \ldots$, are the *mutual impedances* of meshes 1 and 2, p and q, ..., i.e. the impedances common to the designated meshes.

The mutual impedance is defined as follows. Z_{pq} is the p.d. per ampere of I_q in the direction of I_p, and Z_{qp} is the p.d. per ampere of I_p in the direction of I_q. The sign of a mutual impedance depends on the current directions chosen for the meshes concerned. If the network is co-planar (i.e. it can be drawn on a diagram with no cross-over) it is usual to select a single consistent direction—say clockwise—for each mesh current. In such a case the mutual impedances are *negative* because the currents are oppositely directed in the common branches.

3.2.3.2 *Node-voltage equations*

Of the network nodes, one is chosen as a reference node to which all other node voltages are related. The sources are represented by current generators feeding specified currents into their respective nodes and the branches are in terms of admittance Y. Then for the n independent nodes

$$I_a = V_a Y_{aa} + V_b Y_{ab} + \cdots + V_n Y_{an}$$
$$I_b = V_a Y_{ba} + V_b Y_{bb} + \cdots + V_n Y_{bn}$$
$$\vdots \qquad \vdots \qquad \vdots$$
$$I_n = V_a Y_{na} + V_b Y_{nb} + \cdots + V_n Y_{nn}$$

Here Y_{aa}, $Y_{bb}, \ldots,$ Y_{nn} are the *self-admittances* of nodes a, $b, \ldots,$ n, i.e. the sum of the admittances terminating on nodes $a, b, \ldots, n$; and Y_{ab}, $Y_{pq}, \ldots,$ are the *mutual admittances*, those that link nodes a and b, p and $q, \ldots,$ respectively, and which are usually negative.

The mesh-current and node-voltage methods are general and basic; they are applicable to all network conditions. Simplified and auxiliary techniques are applied in special cases.

3.2.3.3 *Techniques*

Steady-state conditions Transient phenomena are absent. For d.c. networks the constant current implies absence of inductive effects, and capacitors (having a constant charge) are equivalent to an open circuit. Only resistance is taken into account, using the Ohm law.

For a.c. networks with sinusoidal current and voltage, complexor algebra, phasor diagrams, locus diagrams and symmetrical components are used, while for a.c. networks with periodic but non-sinusoidal waveforms harmonic analysis with superposition of harmonic components is employed.

Transient conditions Operational forms of stimuli and parameters are used and the solutions are found using Laplace transforms.

3.2.4 Network theorems

Network theorems can simplify complicated networks, facilitate the solution of specific network branches and deal with particular network configurations (such as two-ports). They are applicable to linear networks for which *superposition* is valid, and to any form (scalar, complexor, or operational) of voltage, current, impedance and admittance. In the following, the Ohm and Kirchhoff laws, and the reciprocity and compensation theorems, are basic; star–delta transformation and the Millman theorem are applied to network simplification; and the Helmholtz–Thevenin and Helmholtz–Norton theorems deal with specified branches of a network. Two-ports are dealt with in Section 3.2.5.

3.2.4.1 *Ohm* (*Figure 3.2*(*a*))

For a branch of resistance R or conductance G,

$$I = V/R = VG; \quad V = I/R = I/G; \quad R = V/I = 1/G$$

Summation of resistances R_1, $R_2, \ldots,$ in series or parallel gives

$$\text{Series:} \quad R = R_1 + R_2 + \cdots \text{ or } G = 1/(1/G_1 + 1/G_2 + \cdots)$$
$$\text{Parallel:} \quad R = 1/(1/R_1 + 1/R_2 + \cdots) \text{ or } G = G_1 + G_2 + \cdots$$

The Ohm law is generalised for a.c. and transient cases by $I = V/Z$ or $I(\mathrm{p}) = V(\mathrm{p})/Z(\mathrm{p})$, where p is the operator d/dt.

3.2.4.2 *Kirchhoff* (*Figure 3.4*)

The node and mesh laws are

$$\text{Node:} \quad i_1 + i_2 + \cdots = \Sigma i = 0$$
$$\text{Mesh:} \quad e_1 + e_2 + \cdots = i_1 Z_1 + i_2 Z_2 + \cdots \text{ or } \Sigma e = \Sigma i Z$$

3.2.4.3 *Reciprocity*

If an e.m.f. in branch P of a network produces a current in branch Q, then the same e.m.f. in Q produces the same current in P. The ratio of the e.m.f. to the current is then the *transfer* impedance or admittance.

3.2.4.4 *Compensation*

For given circuit conditions, any impedance Z in a network that carries a current I can be replaced by a generator of zero internal impedance and of e.m.f. $E = -IZ$. Further, if Z is changed by ΔZ, then the effect on all other branches is that which would be produced by an e.m.f. $-I\Delta Z$ in series with the changed branch. By use of this theorem, if the network currents have been solved for given conditions, the effect of a changed branch impedance can be found without re-solving the entire network.

3.2.4.5 *Star–delta* (*Figure 3.5*)

At a given frequency (including zero) a three-branch star impedance network can be replaced by a three-branch delta network, and conversely. For a star Z_a, Z_b, Z_c to be equivalent between terminals AB, BC, CA to a delta Z_1, Z_2, Z_3, it is necessary that

$$Z_a = Z_3 Z_1/Z; \quad Z_1 = Z_a + Z_b + Z_a Z_b/Z_c$$
$$Z_b = Z_1 Z_2/Z; \quad Z_2 = Z_b + Z_c + Z_b Z_c/Z_a$$
$$Z_c = Z_2 Z_3/Z; \quad Z_3 = Z_c + Z_a + Z_c Z_a/Z_b$$

where $Z = Z_1 + Z_2 + Z_3$. The *general* star–mesh conversion concerns the replacement of an n-branch star by a mesh of $\frac{1}{2}n(n-1)$ branches, but *not* conversely; and as the number of mesh branches is greater than the number of star branches when $n > 3$, the conversion is only rarely of use.

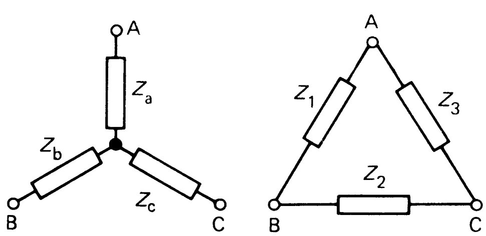


Figure 3.5 Star–delta conversion

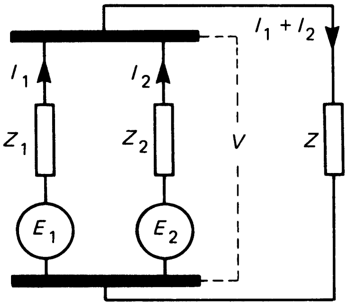


Figure 3.6 The Millman theorem

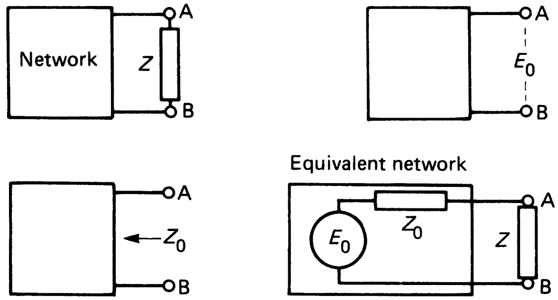


Figure 3.7 The Helmholtz–Thevenin theorem

3.2.4.6 Millman (Figure 3.6)

The Millman theorem is also known as the *parallel-generator* theorem. The common terminal voltage of a number of sources connected in parallel to a common load of impedance Z is $V = I_{sc}Z_p$, where I_{sc} is the sum of the short-circuit currents of the individual source branches and Z_p is the effective impedance of all the branches in parallel, including the load Z. If E_1 and E_2 are the e.m.f.s of two sources with internal impedances Z_1 and Z_2 connected in parallel to supply a load Z, and if I_1 and I_2 are the currents contributed by these sources to the load Z, then their common terminal voltage V must be

$$V = (I_1 + I_2)Z = [(E_1 - V)/Z_1 + (E_2 - V)/Z_2]Z$$

whence

$$V(1/Z + 1/Z_1 + 1/Z_2) = E_1/Z_1 + E_2/Z_2$$

The term in parentheses on the left-hand side of the equation is the effective admittance of all the branches in parallel. The right-hand side of the equation is the sum of the individual source short-circuit currents, totalling I_{sc}. Thus $V = I_{sc}Z_p$. The theorem holds for any number of sources.

3.2.4.7 Helmholtz–Thevenin (Figure 3.7)

The current in any branch Z of a network is the same as if that branch were connected to a voltage source of e.m.f. E_0 and internal impedance Z_0, where E_0 is the p.d. appearing across the branch terminals when they are open-circuited and Z_0 is the impedance of the network looking into the branch terminals with all sources represented by their internal impedance.

In *Figure 3.7*, the network has a branch AB in which it is required to find the current. The branch impedance Z is removed, and a p.d. E_0 appears across AB. With all sources replaced by their internal impedance, the network presents the impedance Z_0 to AB. The current in Z when it is replaced into the original network is

$$I = E_0/(Z_0 + Z) \Leftarrow$$

The whole network apart from the branch AB has been replaced by an equivalent *voltage* source, resulting in the simplified condition of *Figure 3.1(a)*.

3.2.4.8 Helmholtz–Norton

The Helmholtz–Norton theorem is the dual of the Helmholtz–Thevenin theorem. The voltage across any branch Y of a network is the same as if that branch were connected to a current source I_0 with internal shunt admittance Y_0, where I_0 is the current between the branch terminals when short circuited and Y_0 is the admittance of the network looking into the branch terminals with all sources represented by their internal admittance. Then across the terminals AB in *Figure 3.7* the voltage is

$$V = I_0/(Y_0 + Y) \Leftarrow$$

Thus the whole network apart from the branch AB has been replaced by an equivalent *current* source, i.e. the system in *Figure 3.1(b)*.

3.2.5 Two-ports

In power and signal transmission, input voltage and current at one port (i.e. one terminal-pair) yield voltage and current at another port of the interconnecting network. Thus in *Figure 3.8a* voltage source at the input port 1 delivers to the passive network a voltage V_1 and a current I_1. The corresponding values at the output port 2 are V_2 and I_2.

3.2.5.1 Lacour

According to the theorem originated by Lacour, any passive linear network between two ports can be replaced by a two-mesh or T network, and in general no simpler form can be found. Such a result is obtained by iterative star–delta conversion to give the T equivalent; by one more star–delta conversion the Π-equivalent is obtained (*Figure 3.9*). In general, the equivalent networks are asymmetric; in some cases, however, they are symmetric. It can be shown that a passive two-port has the input and output voltages and currents related by

$$V_1 = \boldsymbol{A}V_2 + \boldsymbol{B}I_2 \quad \text{and} \quad I_1 = \boldsymbol{C}V_2 + \boldsymbol{D}I_2$$

where $\boldsymbol{ABCD}$ are the general two-port parameters, constants for a given frequency and with $\boldsymbol{AD} - \boldsymbol{BC} = 1$. The conventions for voltage polarity and current direction are those given in *Figure 3.8*.

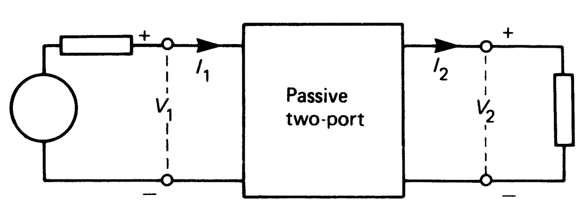


Figure 3.8 Two-port network

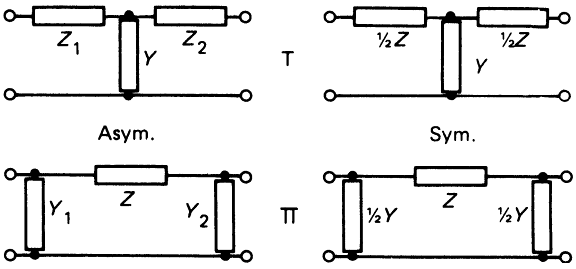


Figure 3.9 T and Π two-ports

3.2.5.2 T *network*

Consider the asymmetric T in *Figure 3.9*. Application of the Kirchhoff laws gives

$$I_1 = I_2 + (V_2 + I_2 Z_2)Y = V_2 Y + I_2(1 + YZ_2)$$
$$V_1 = V_2(1 + YZ_1) + I_2(Z_1 + Z_2 + Z_1 Z_2 Y)$$

Hence in terms of the series and parallel branch components

$$\boldsymbol{A} = 1 + YZ_1$$
$$\boldsymbol{B} = Z_1 + Z_2 + Z_1 Z_2 Y$$
$$\boldsymbol{C} = Y$$
$$\boldsymbol{D} = 1 + YZ_2$$

Multiplication shows that $\boldsymbol{AD} - \boldsymbol{BC} = 1$.

For the symmetric T with $Z_1 = Z_2 = \frac{1}{2}Z$,

$$\boldsymbol{A} = 1 + \tfrac{1}{2}YZ = \boldsymbol{D}; \quad \boldsymbol{B} = Z + \tfrac{1}{4}YZ^2; \quad \boldsymbol{C} = Y$$

3.2.5.3 Π *network*

In a similar way, the general parameters for the asymmetric case are

$$\boldsymbol{A} = 1 + Y_2 Z; \quad \boldsymbol{B} = Z; \quad \boldsymbol{C} = Y_1 + Y_2 + Y_1 Y_2 Z; \quad \boldsymbol{D} = 1 + Y_1 Z$$

which reduce with symmetry to

$$\boldsymbol{A} = 1 + \tfrac{1}{2}YZ = \boldsymbol{D}; \quad \boldsymbol{B} = Z; \quad \boldsymbol{C} = Y + \tfrac{1}{4}Y^2 Z$$

The values of the ***ABCD*** parameters, in matrix form,

$$\begin{pmatrix} \boldsymbol{A} & \boldsymbol{B} \\ \boldsymbol{C} & \boldsymbol{D} \end{pmatrix}$$

are set out in *Table 3.1* for a number of common cases.

3.2.5.4 Characteristic impedance

If the output terminals of a two-port are closed through an impedance $V_2/I_2 = Z_0$, and if the input impedance V_1/I_1 is then also Z_0, the quantity Z_0 is the *characteristic impedance*. Consider a symmetrical two-port ($\boldsymbol{A} = \boldsymbol{D}$) so terminated: if V_1/I_1 is to be Z_0 we have

$$\frac{V_1}{I_1} = \frac{V_2\boldsymbol{A} + I_2\boldsymbol{B}}{V_2\boldsymbol{C} + I_2\boldsymbol{A}} = \frac{V_2(\boldsymbol{A} + \boldsymbol{B}/Z_0)}{I_2(\boldsymbol{A} + \boldsymbol{C}Z_0)} = Z_0\frac{\boldsymbol{A} + \boldsymbol{B}/Z_0}{\boldsymbol{A} + \boldsymbol{C}Z_0}$$

which is Z_0 for $\boldsymbol{B}/Z_0 = \boldsymbol{C}Z_0$. Thus the characteristic impedance is $Z_0 = \sqrt{(\boldsymbol{B}/\boldsymbol{C})}$. The same result is obtainable from the input impedances with the output terminals first open circuited ($I_2 = 0$) giving Z_{oc}, then short circuited ($V_2 = 0$) giving Z_{sc}: thus

$$Z_{oc} = \boldsymbol{A}/\boldsymbol{C}; \quad Z_{sc} = \boldsymbol{B}/\boldsymbol{A}; \quad Z_0 = \sqrt{(Z_{oc}Z_{sc}} = \sqrt{(\boldsymbol{B}/\boldsymbol{C})}$$

3.2.5.5 Propagation coefficient

The parameters ***ABCD*** are functions of frequency, and Z_0 is a complex operator. For the Z_0 termination of a symmetrical two-port (for which $\boldsymbol{A}^2 - \boldsymbol{BC} = 1$) the input/output voltage or current ratio is

$$V_1/V_2 = I_1/I_2 = \boldsymbol{A} + \sqrt{(\boldsymbol{BC})} = \boldsymbol{A} + \sqrt{(\boldsymbol{A}^2 - 1)}$$
$$= \exp(\gamma) = \exp(\alpha + j\beta)$$

The magnitude of V_1 exceeds that of V_2 by the factor $\exp(\alpha)$ and leads it by the angle β, where α is the *attenuation* coefficient, β is the *phase* coefficient and the combination $\gamma = \alpha + j\beta$ is the *propagation* coefficient.

3.2.5.6 Alternative two-port parameters

There are other ways of expressing two-port relationships. For generality, both terminal voltages are taken as *applied* and both currents are *input* currents. With this convention it is necessary to write $-I_2$ for I_2 in the general parameters so far discussed. The mesh-current and node-voltage methods (Section 3.2.4) give $V_1 = I_1 z_{11} + I_2 z_{12}$, etc., and $I_1 = V_1 y_{11} + V_2 y_{12}$, etc., respectively. A further method relates V_1 and I_2 to I_1 and V_2 by hybrid (impedance and admittance) parameters. The four relationships are then obtained as follows:

General

$$\begin{pmatrix} V_1 \\ I_1 \end{pmatrix} = \begin{pmatrix} \boldsymbol{A} & \boldsymbol{B} \\ \boldsymbol{C} & \boldsymbol{D} \end{pmatrix} \begin{pmatrix} V_2 \\ -I_2 \end{pmatrix}$$

Impedance

$$\begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \begin{pmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{pmatrix} \begin{pmatrix} I_1 \\ I_2 \end{pmatrix}$$

Admittance

$$\begin{pmatrix} I_1 \\ I_2 \end{pmatrix} = \begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \end{pmatrix}$$

Hybrid

$$\begin{pmatrix} V_1 \\ I_2 \end{pmatrix} = \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix} \begin{pmatrix} I_1 \\ V_2 \end{pmatrix}$$

If the networks are passive, then $z_{12} = z_{21}$, $y_{12} = y_{21}$ and $h_{12} = -h_{21}$. If, in addition, the networks are symmetrical, then $\boldsymbol{A} = \boldsymbol{D}$, $z_{11} = z_{22}$ and $y_{11} = y_{22}$. If the networks are active (i.e. they contain sources), then reciprocity does not apply and there is no necessary relation between the terms of the 2×2 matrix.

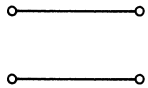
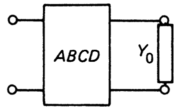
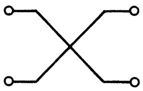
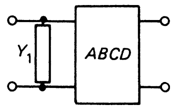
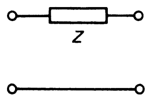
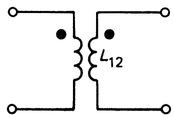
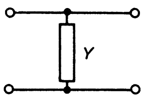
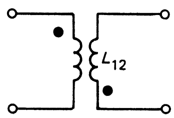
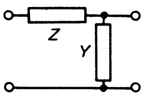
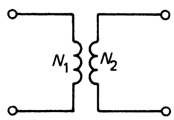
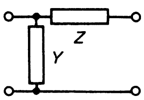
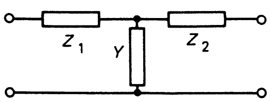
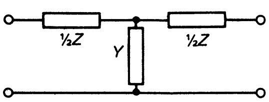
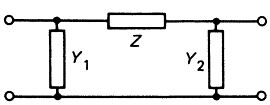
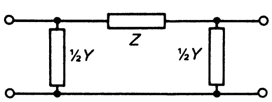
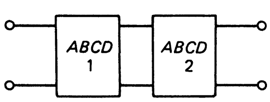
3.2.6 Network topology

In multibranch networks the solution process is aided by representing the network as a graph of nodes and interconnections. The topology is the scheme of interconnections. A network is planar if it can be drawn on a closed spherical (or plane) surface without cross-overs. A non-planar network cannot be so drawn: a single cross-over can be eliminated if the network is drawn on a more complicated surface resembling a doughnut, and more cross-overs require closed surfaces with more holes.

The nomenclature employed in topology is as follows.

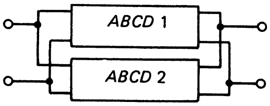
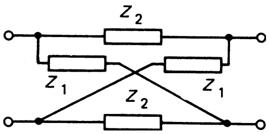
Graph A diagram of the network showing all the nodes, with each branch represented by a plain line.
Tree Any arrangement of branches that connects all nodes together without forming loops. A *tree branch* is one branch of such a tree.
Link A branch that, added to a tree, completes a closed loop.
Tie set A loop of branches with one a link and the others tree branches.
Cut set A set of branches comprising one tree branch, the other branches being tree links. A cut set dissociates two main portions of a network in such a way that replacing any one element destroys the dissociation.

Table 3.1 General ***ABCD*** two-port parameters

Network	*Matrix*	*Network*	*Matrix*
	Direct connection $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$		Loaded network $\begin{pmatrix} \boldsymbol{A} + \boldsymbol{B}Y_0 & \boldsymbol{B} \\ \boldsymbol{C} + \boldsymbol{D}Y_0 & \boldsymbol{D} \end{pmatrix}$
	Cross-connection $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$		Shunted network $\begin{pmatrix} \boldsymbol{A} & \boldsymbol{B} \\ \boldsymbol{C} + \boldsymbol{A}Y_1 & \boldsymbol{D} + \boldsymbol{B}Y_1 \end{pmatrix}$
	Series impedance $\begin{pmatrix} 1 & Z \\ 0 & 1 \end{pmatrix}$		Mutual inductance $\begin{pmatrix} 0 & -j\omega L_{12} \\ -1/j\omega L_{12} & 0 \end{pmatrix}$
	Shunt admittance $\begin{pmatrix} 1 & 0 \\ Y & 1 \end{pmatrix}$		Mutual inductance $\begin{pmatrix} 0 & -j\omega L_{12} \\ 1/j\omega L_{12} & 0 \end{pmatrix}$
	L network $\begin{pmatrix} 1 + YZ & Z \\ Y & 1 \end{pmatrix}$		Ideal transformer $\begin{pmatrix} N_1/N_2 & 0 \\ 0 & N_2/N_1 \end{pmatrix}$
	L network $\begin{pmatrix} 1 & Z \\ Y & 1 + YZ \end{pmatrix}$		
	T network $\begin{pmatrix} 1 + YZ_1 & Z_1 + Z_2 + YZ_1Z_2 \\ Y & 1 + YZ_2 \end{pmatrix}$		
	Symmetrical T network $\begin{pmatrix} 1 + YZ/2 & Z(1 + YZ/4) \\ Y & 1 + YZ/2 \end{pmatrix}$		
	Π network $\begin{pmatrix} 1 + Y_2Z & Z \\ Y_1 + Y_2 + Y_1Y_2Z & 1 + Y_1Z \end{pmatrix}$		
	Symmetrical Π network $\begin{pmatrix} 1 + YZ/2 & Z \\ Y(1 + YZ/4) & 1 + YZ/2 \end{pmatrix}$		
	Cascaded networks $\begin{pmatrix} \boldsymbol{A}_1\boldsymbol{A}_2 + \boldsymbol{B}_1\boldsymbol{C}_2 & \boldsymbol{A}_1\boldsymbol{B}_2 + \boldsymbol{B}_1\boldsymbol{D}_2 \\ \boldsymbol{A}_2\boldsymbol{C}_1 + \boldsymbol{C}_2\boldsymbol{D}_1 & \boldsymbol{B}_2\boldsymbol{C}_1 + \boldsymbol{D}_1\boldsymbol{D}_2 \end{pmatrix}$		

cont'd

Table 3.1 *(continued)*

Network	*Matrix*
	Parallel networks $\begin{pmatrix} (A_1B_2 + A_2B_1)/(B_1 + B_2) & B_1B_2/(B_1 + B_2) \\ C_1 + C_2 + \dfrac{(A_1 - A_1)(D_1 - D_2)}{B_1 + B_2} & \dfrac{B_1D_2 + B_2D_1}{B_1 + B_2} \end{pmatrix}$
	Symmetrical lattice network $\begin{pmatrix} (Z_1 + Z_2)/(Z_1 - Z_2) & 2Z_1Z_2/(Z_1 - Z_2) \\ 2/(Z_1 - Z_2) & (Z_1 + Z_2)/(Z_1 - Z_2) \end{pmatrix}$

Before setting up the equations for network solution, some guide is necessary in forming the proper number of independent equations. Given the network (*a*) in *Figure 3.10*, the first step is to draw the graph (*b*). Two of its possible trees are shown in (*c*). The trees are then used to set up the equations.

3.2.6.1 Network equations

Voltage The network diagram for the upper tree in *Figure 3.10*(*c*) is drawn in (*d*). Specifying the tree-branch voltages specifies also the voltages across the links. It is convenient to choose r as a reference node, leaving $n = 6$ independent nodes requiring $n = 6$ voltage equations.

Current A tree has no closed paths. As the links are added with specified currents, each creates one loop. Then the sum of the links m is the number of currents to be evaluated. For a network of b branches and n independent nodes, the number of independent meshes is $m = b - n$.

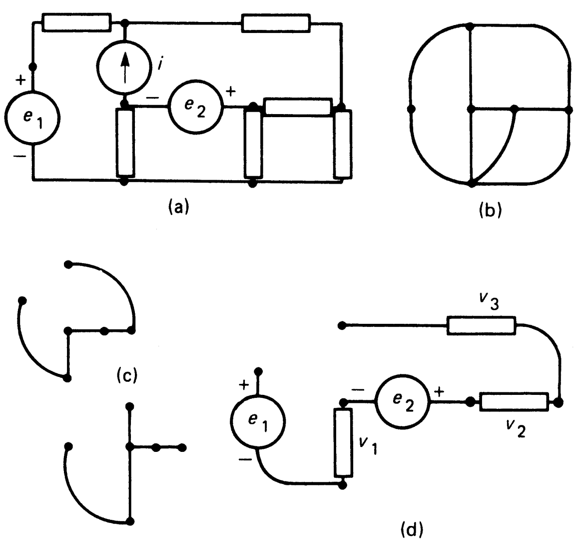


Figure 3.10 Network topology

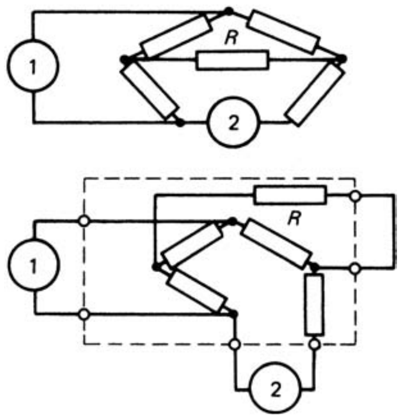


Figure 3.11 Conversion to a passive network

3.2.6.2 Ports

It is often helpful to place the sources outside the network and to regard their connections to the (now passive) remainder as ports. Again, branches of interest can be taken outside and used to terminate ports, as in *Figure 3.11*.

A multiport network (*Figure 3.12*) has the following characteristic definitions:

(a) All ports but one are open-circuited: a voltage V_1 is applied to port 1 and a current I_1 flows into it. Then V_1/I_1 is the *open-circuit* (*o.c.*) *driving-point impedance*

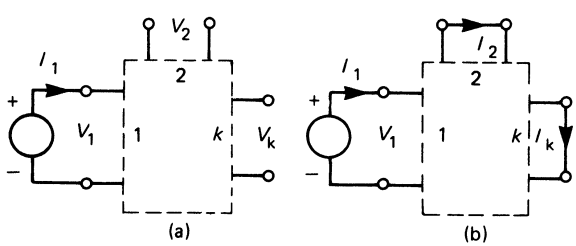


Figure 3.12 A multiport network

at port 1, V_k/I_1 is the *o.c. transfer impedance* from port 1 to port k, and V_k/V_1 is the *o.c. voltage ratio* of port k to port 1.

(b) All ports but one are short circuited: a current I_1 (requiring a voltage V_1) is fed in at port 1. Then I_1/V_1 is the *short-circuit* (*s.c.*) *driving-point admittance* at port 1, I_k/V_1 is the *s.c. transfer admittance* from port 1 to port k, and I_k/I_1 is the *s.c. current ratio* of port k to port 1.

3.2.7 Steady-state d.c. networks

The steady state implies that energy storage in electric and magnetic fields does not change, and only the resistance is significant. In *Figure 3.13* a source of constant e.m.f. E and internal resistance r provides a current I at terminal voltage V to a network represented by an equivalent resistance R. On open circuit ($R=\infty$), $I=0$ and $V=E$. As R is reduced the source provides a current $I=E/(R+r)=(E-V)/r$. The greatest output power $P=VI$ occurs for the condition $R=r$; further reduction of R reduces the network power, down to a short-circuit condition for $R=0$ and $V=0$ when the source power is dissipated entirely in r. The maximum-power condition is utilised only with sources whose power capability is very small.

3.2.8 Steady-state a.c. networks

An a.c. flows alternately in the specified *positive* direction and then in the *negative* direction in a circuit, repeating this cycle continuously. A graph of current or voltage to a time base shows the *waveform* as a succession of *instantaneous values*. In general there will be a maximum or *peak* value in both positive and negative half-periods where the current or voltage is greatest. The time for one complete *cycle* is the *period* T. The number of periods per second is the *frequency* $f=1/T$.

An a.c. is produced by an alternating voltage. Two such quantities may have a difference of *phase*, to which a precise meaning can be given only when the quantities are both sinusoidal functions of time.

3.2.8.1 Root-mean-square (r.m.s.) value

The numerical value assigned to an a.c. or voltage is normally defined in terms of mean power in a pure resistor. An *a.c.* of 1 A is that which produces heat energy at the same mean rate as a direct current of 1 A in the same non-reactive resistor. If i is the instantaneous value of an a.c. in a pure resistance R, the heat developed in a time element dt is $dw=i^2R\,dt$. The mean rate (i.e. the mean power) over a complete period T is

$$P=(1/T)\int_0^T dw=(1/T)\int_0^T i^2R\,dt=I^2R$$

and I is the r.m.s. value of the current. An *alternating voltage* is defined in a similar way; the instantaneous power is v^2/R, and the mean is V^2/R where V is the square root of the mean v^2.

In some cases the *peak* or the *mean* value of the current or voltage waveform is more significant, particularly with asymmetric, pulse or rectified waveforms. The value to be understood by the term 'mean' is then obvious. In the case of a symmetrical wave, the *half-period mean* value is intended, as the mean over a complete period is zero. *Table 3.2* gives the mean and r.m.s. values for a number of typical waveforms, together with the values of

$$\text{Form factor } K_f=\text{r.m.s./mean}$$
$$\text{Peak factor } K_p=\text{peak/r.m.s.}$$

Table 3.2 Values of alternating quantities (peak = a)

Waveform	*r.m.s.*	*Mean*	K_f	K_p
Sinusoidal	$a/\sqrt{2}$	$a(2/\pi)$	1.11	1.41
Half-wave rectified sine	$a/2$	a/π	1.57	2.0
Full-wave rectified sine	$a/\sqrt{2}$	$a(2/\pi)$	1.11	1.41
Rectangular	a	a	1.0	1.0
Triangular	$a/\sqrt{3}$	$a/2$	1.16	1.73

The techniques developed for the solution of steady-state a.c. networks depend on the waveform. One technique applies to purely sinusoidal quantities, another to periodic but non-sinusoidal waveforms. In each case the network is assumed to be linear so that the principle of superposition is valid.

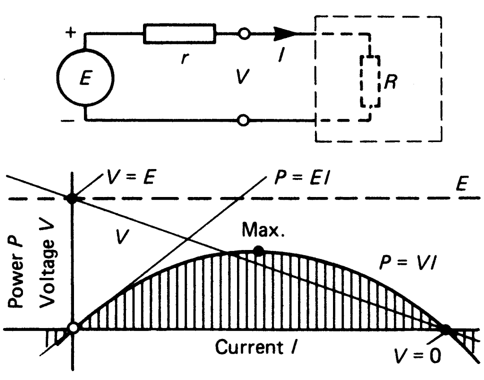


Figure 3.13 A d.c. system

3.2.9 Sinusoidal alternating quantities

For pure sinusoidal waveforms, a current can be expressed as a function of time, $i=i_m\sin(2\pi ft)=i_m\sin(\omega t)$, completing f cycles in 1 s with a period $T=1/f$. The quantity $2\pi f$ is contracted to ω, the *angular frequency*. The sine-wave shape has the advantages that (i) it is mathematically simple and its integral and differential are both cosinusoidal, (ii) it is a waveform desirable for power generation, transmission and utilisation, and (iii) it lends itself to phasor and complexor representation.

The graph of a sinusoidal current or voltage of frequency f can be plotted to a time-angle base ωt by use of trigonometric tables. Alternatively it can be represented by the projection of a line of length equal to the peak value and rotating counter-clockwise at angular speed ω about one end O. A *stationary* line can represent the sine wave, particularly in relation to other sine waves of the same frequency but 'out of step'. Two such waves, say v and i with peak values v_m and i_m, respectively, can be written

$$v=v_m\sin\omega t \quad\text{and}\quad i=i_m\sin(\omega t-\phi)$$

and drawn as in *Figure 3.14*, the phase difference or phase angle between them being ϕ rad. Then two lines, OA and OB, having an angular displacement ϕ, can represent the two waves in peak magnitude and relative time phase.

Although developed from rotating lines of peak-value length, it is more convenient to change the scale and treat the lengths as r.m.s. values. The processes of addition and

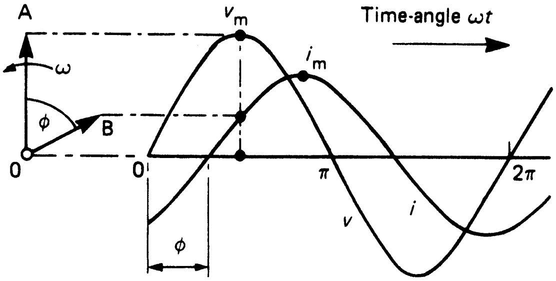


Figure 3.14 Phasors

subtraction of r.m.s. values are performed as if the lines were co-planar vector forces in mechanics. Physically, however, the lines are not vectors: they substitute for scalar quantities, alternating sinusoidally with time. They are termed *phasors*. Certain associated quantities, such as impedance, admittance and apparent power, can also be represented by directed lines, but as they are not sinusoids they are termed *complexors* or *complex operators*. Both phasors and complexors can be dealt with by application of the theory of complex numbers. The definitions concerned are listed below.

Complexor A generic term for a non-vector quantity expressed as a complex number.
Phasor A complexor (e.g. voltage or current) derived from a time-varying sinusoidal quantity and expressed as a complex number.
Complex operator A complexor derived for the ratio of two phasors (e.g. impedance and admittance); or a complexor which, operating on a phasor, gives another phasor (e.g. $V = IZ$, in which V and I are phasors, but Z is a complex operator).

3.2.9.1 Complexor algebra

The four arithmetic processes for complexors are applications of the theory of complex numbers. Complexor $\boldsymbol{a}$ in *Figure 3.15* can be expressed by its magnitude a and its angle θ with respect to an arbitrary 'datum' direction (here taken as horizontal) as the simple *polar* form $\boldsymbol{a} = a \angle \theta$. Alternatively it can be written as $\boldsymbol{a} = p + \mathrm{j}q$, the *rectangular* form, in terms of its projection p on the datum and q on a quadrature axis at right angles thereto: q (as a scalar magnitude along the datum) is rotated counter-clockwise by angle $\frac{1}{2}\pi$ rad (90°) by the operator j. Two successive operations by j (written as j^2) give a rotation of π rad (180°), making the original $+q$ into $-q$, in effect a multiplication by -1. Three operations (j^3) give $-\mathrm{j}q$ and four give $+q$. Thus any complexor can be located in the complex datum–quadrature plane. Further obvious forms are the *trigonometric*, $\boldsymbol{a} = a(\cos\theta + \mathrm{j}\sin\theta)$, and the *exponential*, $\boldsymbol{a} = a\exp(\mathrm{j}\theta)$. Summarising, the four descriptions are:

Polar: $\boldsymbol{a} = a \angle \theta$
Rectangular: $\boldsymbol{a} = p + \mathrm{j}q$

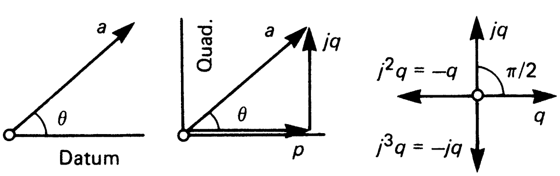


Figure 3.15 Complexors

Exponential: $\boldsymbol{a} = a\exp(\mathrm{j}\theta)$
Trigonometric: $\boldsymbol{a} = a(\cos\theta + \mathrm{j}\sin\theta)$

where $a = \sqrt{(p^2 + q^2)}$ and $\theta = \arctan(q/p)$.

Consider complexors $\boldsymbol{a} = p + \mathrm{j}q = a \angle \alpha$ and $\boldsymbol{b} = r + \mathrm{j}s = b \angle \beta$. The basic manipulations are:

Addition $\boldsymbol{a} + \boldsymbol{b} = (p + r) + \mathrm{j}(q + s)$

Subtraction $\boldsymbol{a} - \boldsymbol{b} = (p - r) + \mathrm{j}(q - s)$

Multiplication The exponential and polar forms are more direct than the rectangular or trigonometric:

$$\boldsymbol{ab} = (pr - qs) + \mathrm{j}(qr + ps)$$
$$= ab\exp[\mathrm{j}(\alpha + \beta)] = ab \angle (\alpha + \beta)$$

Division Here also the angular forms are preferred:

$$\boldsymbol{a}/\boldsymbol{b} = [(pr + qs) + \mathrm{j}(qr - ps)]/(r^2 + s^2)$$
$$= (a/b)\exp[\mathrm{j}(\alpha - \beta)] = (a/b) \angle (\alpha - \beta)$$

Conjugate The conjugate of a complexor $a = p + \mathrm{j}q = a \angle \alpha$ is $\boldsymbol{a}^* = p - \mathrm{j}q = a \angle (-\alpha)$, the quadrature component (and therefore the angle) being reversed. Then

$$\boldsymbol{ab}^* = ab \angle (\alpha - \beta)$$
$$\boldsymbol{a}^*\boldsymbol{b} = ab \angle (\beta - \alpha)$$
$$\boldsymbol{a}^*\boldsymbol{a} = \boldsymbol{aa}^* = a^2 = p^2 + q^2$$

The last expression is used to 'rationalise' the denominator in the complexor division process.

3.2.9.2 Impedance and admittance operators

Sinusoidal voltages and currents can be represented by phasors in the expressions $V = IZ = I/Y$ and $I = VY = V/Z$. Current and voltage phasors are related by multiplication or division with the complex operators Z and Y. Series resistance R and reactance $\mathrm{j}X$ can be arranged as a right-angled triangle of hypotenuse $Z = \sqrt{(R^2 + X^2)}$ and the angle between Z and R is $\theta = \arctan(X/R)$. The relation between Z and Y for the same series network elements with $Z = R + \mathrm{j}X$ is

$$Y = \frac{1}{Z} = \frac{1}{R + \mathrm{j}X} = \frac{R - \mathrm{j}X}{(R + \mathrm{j}X)(R - \mathrm{j}X)} = \frac{R - \mathrm{j}X}{R^2 + X^2}$$
$$= R/Z^2 - \mathrm{j}(X/Z^2) = G - \mathrm{j}B$$

where G and B are defined in terms of R, X and Z. The *series* components R and X become *parallel* branches in Y, one a pure conductance, the other a pure susceptance. Further, a positive-angled impedance has, as inverse equivalent, a negative-angled admittance (*Figure 3.16*).

The impedance and phase angle of a number of circuit combinations are given in *Table 3.3*.

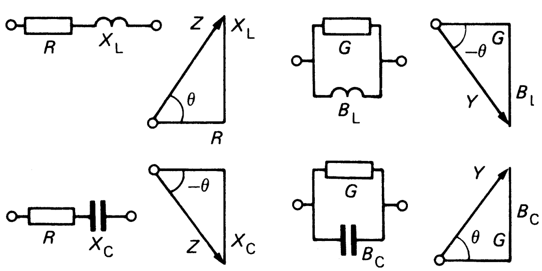
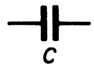
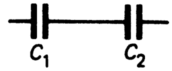
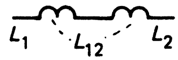
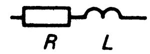
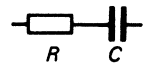
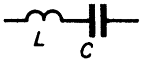
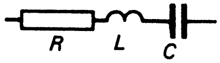
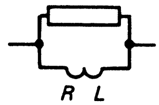
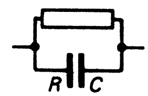
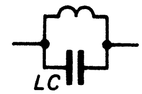


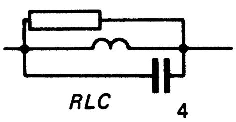
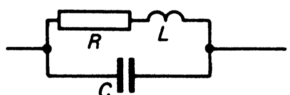
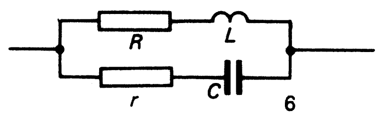
Figure 3.16 Impedance and admittance triangles

Table 3.3 Impedance of network elements at angular frequency ω (rad/s)

Impedance:	$Z = R + jX = \lvert Z\rvert \angle \theta$	$\lvert Z\rvert = \sqrt{(R^2+X^2)}$	$\theta = \arctan(X/R)$
Admittance:	$Y = 1/Z = \lvert Y\rvert \angle(-\theta)$	$\lvert Y\rvert = \sqrt{[(R/Z^2)^2 + (X/Z^2)^2]}$	$\theta = -\arctan(X/R)$

							
Z:	R	$1/j\omega C$	$j\omega L$	$(C_1+C_2)/j\omega C_1C_2$	$j\omega(L_1+2L_{12})$	$R+j\omega L$	$R+1/j\omega C$
θ:	0	$-\pi/2$	$+\pi/2$	$-\pi/2$	$+\pi/2$	$\arctan(\omega L/R)$	$-\arctan(j\omega CR)$

	 1	 2			 3
Z:	$j(\omega L - 1/\omega C)$	$R + j(\omega L - 1/\omega C)$	$\omega LR\dfrac{\omega L + jR}{R^2+\omega^2L^2}$	$R\dfrac{1-j\omega CR}{1+\omega^2C^2R^2}$	$\dfrac{j\omega L}{1-\omega^2LC}$
θ:	$\pm\pi/2$	$\arctan[(\omega L - 1/\omega C)/R]$	$\arctan(R/\omega L)$	$-\arctan(\omega CR)$	$\pm\pi/2$

	 4	 5	 6
Z:	$\dfrac{1/R - j(\omega C - 1/\omega L)}{(1/R)^2 + (\omega C - 1/\omega L)^2}$	$\dfrac{R + j\omega[L(1-\omega^2LC) - CR^2]}{(1-\omega^2LC)^2 + \omega^2C^2R^2}$	$\dfrac{A + jB}{(R+r)^2 + (\omega L - 1/\omega C)^2}$ $A = Rr(R+r) + \omega^2L^2r + R/\omega^2C^2$ $B = \omega Lr^2 - R^2/\omega C - (L/C)(\omega L - 1/\omega C)$
θ:	$-\arctan[R(\omega C - 1/\omega L)]$	$\arctan\{\omega[L(1-\omega^2 LC) - CR^2]/R\}$	$\arctan(B/A)$

Resonance conditions for LC networks numbered 1–6 above, for $\omega = \omega_0 = 1/\sqrt{(LC)}$:
(1) $|Z| = 0$, $\theta = 0$; (2) $|Z| = R$, $\theta = 0$; (3) $|Z| = \infty$, $\theta = 0$; (4) $|Z| = R$, $\omega = 0$;
(5) $|Z| = L/CR$, $\theta = -\arctan(\omega CR)$ for $R \ll \omega L$; (6) $|Z| = R$ (const.) for $R = r = \sqrt{(L/C)}$

Impedance and admittance loci If the characteristics of a device or a circuit can be expressed in terms of an equivalent circuit in which the impedances and/or admittances vary according to some law, then the current taken for a given applied voltage (or the voltage for a given current) can be obtained graphically by use of an admittance or impedance locus diagram.

In *Figure 3.17(a)*, let OP represent an impedance $Z = R + jX$ and OQ the corresponding admittance $Y = G - jB$. Point Q is obtained from P by finding first the *geometric* inverse point Q′ such that OQ′ = 1/OP to scale, and then reflecting OQ′ across the datum line to give OQ and thus a reversed angle $-\theta$, a process termed *complexor* inversion. If Z has successive values $Z_1, Z_2, \ldots$, on the impedance locus, the corresponding admittances $Y_1, Y_2, \ldots$, lie on the admittance locus. The inversion process may be point-by-point, but in many cases certain propositions can reduce the labour:

(1) *Inverse of a straight line*—the geometric inverse of a straight line AB about a point O not on the line is a circle passing through O with its centre M on the perpendicular OC from O to AB (*Figure 3.17(b)*). Then A′ is the geometric inverse of A, B′ of B, etc.; also, A is the inverse of A′, B of B′, etc.

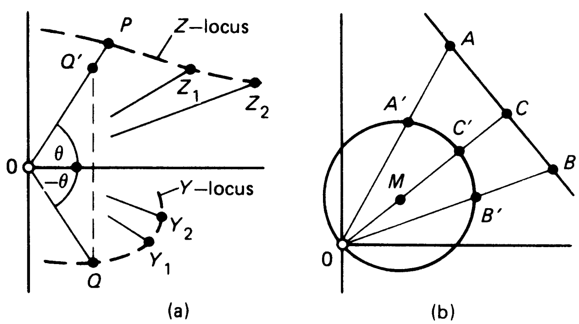


Figure 3.17 Inversion

(2) *Inverse of a circle*—from the foregoing, the geometric inverse of a circle about a point O on its circumference is a straight line. If, however, O is not on the circumference, the inverse is a second circle between the same tangents; but the distances OM and OM′ from the origin O to the centres M and M′ of the circles are not inverses of each other.

The choice of scales arises in the inversion process: for example, the inverse of an impedance $Z = 50\ \angle\ 70°\ \Omega$ is $Y = 0.02\ \angle\ (-70°)$ S. It is usually possible to decide on a scale by taking a salient feature (such as a circle diameter) as a basis.

3.2.9.3 Power

The instantaneous power delivered to a load is the product of the instantaneous voltage v and current i. Let $v = v_m \sin \omega t$ and $i = i_m \sin(\omega t - \phi)$ as in *Figure 3.18*(*a*); then the instantaneous power is

$$p = \tfrac{1}{2} v_m i_m [\cos\phi - \cos(2\omega t - \phi)]$$

This is a quantity fluctuating at angular frequency 2ω with, in general, excursions into negative power (i.e. that returned by the load to the source). Over an integral number of periods the mean power is

$$P = \tfrac{1}{2} v_m i_m \cos\phi = VI \cos\phi$$

where V and I are r.m.s. values.

Now resolve i into the active and reactive components

$$i_p = (i_m \cos\phi)\sin\omega t \text{ and } i_q = (i_m \sin\phi)\sin(\omega t - \tfrac{1}{2}\pi)$$

as in *Figure 3.18*(*b*); then the instantaneous power can be written

$$p = (v_m(i_m \cos\phi)\sin^2\omega t - v_m(i_m \sin\phi)\sin\omega t \cos\omega t$$

Over a whole number of periods the average of the first term is

$$P = \tfrac{1}{2} v_m i_m \cos\phi = VI \cos\phi$$

giving the *average* rate of energy transfer from source to load. The second term is a double-frequency sinusoid of average value zero, the energy flow changing direction rhythmically between source and load at a *peak* rate

$$Q = \tfrac{1}{2} v_m i_m \sin\phi = VI \sin\phi$$

The power conditions thus summarise to the following:

Active power P The *mean* of the instantaneous power over an integral number of periods giving the mean rate of energy transfer from source to load in watts (W).

Reactive power Q The *maximum* rate of energy interchange between source and load in reactive volt-amperes (var).

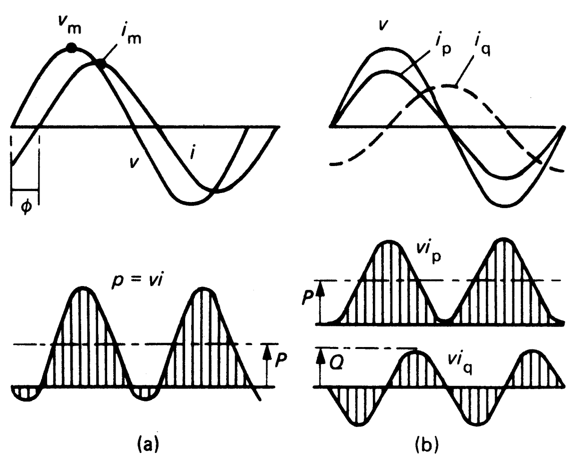


Figure 3.18 Active and reactive power

Apparent power S The product of the r.m.s. voltage and current in volt-amperes (V-A).

Both P and Q represent real power. The apparent power S is not a power at all, but is an arbitrary product VI. Nevertheless, because of the way in which P and Q are defined, we can write

$$P^2 + Q^2 = (VI)^2(\cos^2\phi + \sin^2\phi) = (VI)^2$$

whence $S = \sqrt{(P^2 + Q^2)}$, a convenient combination of mean active power with peak power circulation.

Complex power The active and reactive powers can be determined for voltage and current phasors by

$$S = P \pm jQ = VI^* \quad \text{or} \quad S = V^*I$$

using the conjugate of either I or V.

Power factor The ratio of active to apparent power, $P/S = \cos\phi$ for sinusoidal conditions.

3.2.9.4 Resonance

A condition of resonance occurs when the load contains two forms of energy-storing element (L and C) such that, at the frequency of operation, the two energies are equal. The reactive power requirements are then satisfied internally, as the inductor releases energy at the rate that the capacitor requires it. The source supplies only the active power demand of the energy-dissipating load components, the load externally appearing to be purely resistive.

Acceptor resonance The series *RLC* circuit in *Figure 3.19*(*a*) has, at angular frequency ω, the impedance $Z = R + jX$, where X is $\omega L - 1/\omega C$, which for $\omega = \omega_0 = 1/\sqrt{(LC)}$ is zero. The impedance is then $Z = R$ and the input current has a maximum $I_0 = V/R$, conditions of *acceptor* resonance. Internally, large voltages appear across the reactive components, *viz*.

$$V_L = I_0 \omega L = V\omega_0(L/R) \text{ and } V_C = I_0(1/\omega_0 C) = V/\omega_0(CR)$$

The terms L/R and $1/CR$ are the time constants of the reactive elements, and $\omega_0 L/R$ is the Q value of a practical inductor of

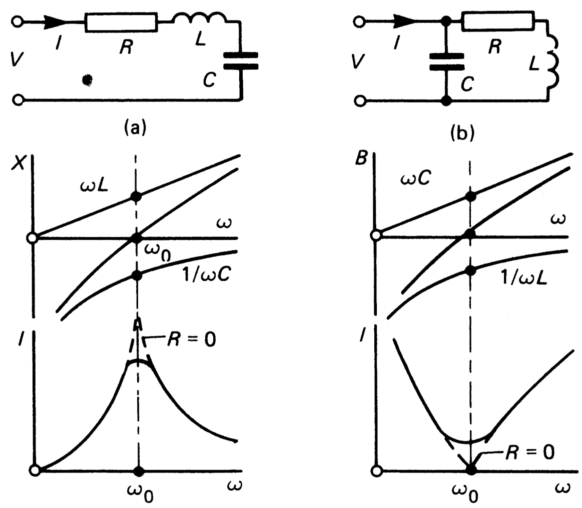


Figure 3.19 Resonance

inductance L and loss-resistance R. The Q value may be large (e.g. 100) for resonance at a high frequency.

Rejector resonance This occurs in a parallel combination of L and C, the energies circulating around the closed LC loop. If in *Figure 3.19(b)* the resistance R is zero, the terminal input admittance vanishes at angular frequency $\omega_0 = 1/\sqrt{(LC)}$, with $\omega_0 C = 1/\omega_0 L$ and an input susceptance $B = 0$. Where the circuit contains resistance R the resonance conditions are less definite. Three possible criteria are: (i) $\omega_0 = 1/\sqrt{(LC)}$, (ii) the input admittance is a minimum, and (iii) the input admittance is purely conductive. All three criteria are satisfied simultaneously in the simple acceptor circuit, but differ in rejector conditions; however, where resonance is an intended property of the circuit, the differences are small.

Some expressions for resonance are given for six circuit arrangements in *Table 3.3*.

3.2.10 Non-sinusoidal alternating quantities

Periodic but non-sinusoidal currents occur: (i) with non-sinusoidal e.m.f. sources, (ii) with sinusoidal sources applied to non-linear loads, and (iii) with any combination of (i) and (ii).

3.2.10.1 Fourier series

Any univalued periodic waveform can be represented as a summation of sine waves comprising a *fundamental*, where frequency is that of the periodic occurrence, and a series of *harmonic* waves of frequency 2, 3, ..., n times that of the fundamental. The Fourier series for a periodic function $y = f(x)$ takes either of the following equivalent forms:

$$(1)\ y = c_0 + c_1 \sin(x + \alpha_1) + c_2 \sin(2x + \alpha_2) + \cdots$$

$$(2)\ y = c_0 + a_1 \cos x + a_2 \cos 2x + \cdots + a_n \cos nx + b_1 \sin x + b_2 \sin 2x + \cdots + b_n \sin nx$$

where c_0 is a constant, $c_n = \sqrt{(a_n{}^2 + b_n{}^2)}$ and $\alpha_n =$ arctan (a_n/b_n). The coefficients of the terms are given by

$$c_0 = (1/2\pi)\int_0^{2\pi} f(x)\,\mathrm{d}x = \text{mean of the wave over one period}$$

$$a_n = (1/\pi)\int_0^{2\pi} f(x) \cos nx\,\mathrm{d}x$$

$$b_n = (1/\pi)\int_0^{2\pi} f(x) \sin nx\,\mathrm{d}x$$

These can be evaluated mathematically for simple cases. The work may sometimes be reduced by inspection: thus $c_0 = 0$ for a wave symmetrical about the baseline; or only odd-order harmonics may be present.

3.2.10.2 Analysis

The series for a range of mathematically tractable waveforms are given in *Table 1.10*. For experimentally derived waveforms there are several methods, but none yields the amplitude of higher order harmonics without considerable labour, unless a computer program is available.

A particular harmonic, say the nth, may be found by superimposing n copies of the wave, displaced relatively by $2\pi/n$, $4\pi/n, \ldots$, and adding the corresponding ordinates. The result is a wave of frequency n times that of the harmonic sought (with the addition, however, of harmonics of orders kn where k is an integer). The method gives also the phase angle α_n.

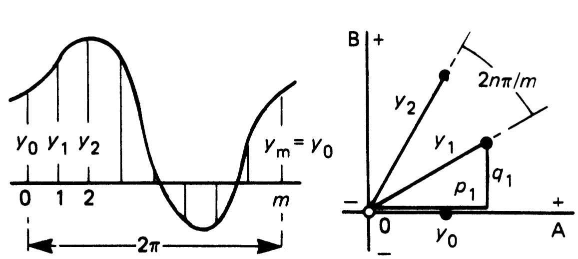


Figure 3.20 Graphical harmonic analysis

A semi-graphical method is shown in *Figure 3.20*. The base of a complete period 2π is divided into m parts, the corresponding ordinates being $y_0, y_1, y_2, \ldots, y_m$. Construct axes OA and OB; set out the radii y_0 to y_m $(=y_0)$ at angles 0, $2n\pi/m$, $4n\pi/m, \ldots$, from the axis OA. Then project the extremities horizontally (p) and vertically (q), and take the sum of the two sets of projections with due regard to their sign. Then for the nth harmonic

$$a_n = \frac{2}{\pi}\sum_0^{m-1} p \quad \text{and} \quad b_n = \frac{2}{\pi}\sum_0^{m-1} q$$

The labour is reduced if $2\pi/m$ is a simple fraction of 2π, for then some groups of radii are coincident.

3.2.10.3 Power

The r.m.s. value of a current

$$i = I_0 + i_1 \sin(\omega t + \alpha_1) + i_2 \sin(2\omega t + \alpha_2) + \cdots$$

is obtained from the square root of the average squared value, resulting in

$$I = \sqrt{(I_0^2 + \tfrac{1}{2} i_1^2 + \tfrac{1}{2} i_2^2 + \cdots)} = \sqrt{(I_0^2 + I_1^2 + I_2^2 + \cdots)}$$

where $I_1 = i_1/\sqrt{2}$, $I_2 = i_2/\sqrt{2}$, etc., are the r.m.s. values of the individual harmonic components. The r.m.s. voltage is obtained in a similar way.

Power The instantaneous power p in a circuit with an applied voltage

$$v = v_1 \sin(\omega t + \alpha_1) + v_2 \sin(2\omega t + \alpha_2) + \cdots$$

producing a current

$$i = i_1 \sin(\omega t + \alpha_1 - \phi_1) + i_2 \sin(2\omega t + \alpha_2 - \phi_2) + \cdots$$

is the product vi: this includes (i) a series of the form

$$v_n i_n \sin(n\omega t + \alpha_n) \sin(n\omega t + \alpha_n - \phi_n)$$

all terms of which have a fundamental-period average $\frac{1}{2} v_n i_n \cos \phi_n$; and (ii) a series of the form

$$v_p i_q \sin(p\omega t + \alpha_p) \sin(q\omega t + \alpha_q - \phi_q)$$

which, over a fundamental period, averages zero. Power is circulated by a voltage and a current of different frequencies, but the circulation averages zero. The mean (active) power is therefore

$$P = \tfrac{1}{2} v_1 i_1 \cos \phi_1 + \tfrac{1}{2} v_2 i_2 \cos \phi_2 + \cdots + \tfrac{1}{2} v_n i_n \cos \phi_n + \cdots$$
$$= V_1 I_1 \cos \phi_1 + V_2 I_2 \cos \phi_2 + \cdots + V_n I_n \cos \phi_n + \cdots$$

where the capital letters denote component r.m.s. values. Thus the harmonics contribute power separately.

Power factor The ratio of the active power P to the apparent power S is

$$P/S = (V_1 I_1 \cos\phi_1 + \cdots + V_n I_n \cos\phi_n + \cdots)/VI$$

This may be less than unity even with all phase angles zero if the ratio V_n/I_n is not the same for each component. Where the applied voltage is a *pure sinusoid* of fundamental frequency there can be no harmonic powers; the active power is $P = V_1 I_1 \cos\phi_1$. Then

$$P/S = (V_1 I_1 \cos\phi_1)/V_1 I = (I_1/I)\cos\phi_1$$

where $I_1/I = I_1/\sqrt{(I_1^2 + I_2^2 + \cdots + I_n^2)} = \delta$, the *distortion factor*. The overall power factor is consequently $\delta\cos\phi_1$. This is typical of circuits containing non-linear elements.

3.2.11 Three-phase systems

A symmetrical m-phase system has m source e.m.f.s, all of the same waveform and frequency, and displaced $2\pi/m$ rad or $1/m$ period in time; m is most commonly 3, but is occasionally 6, 12 or 24.

Symmetric three-phase system In *Figure 3.21*(*a*) the symmetric sinusoidal three-phase system has source e.m.f.s in phases A, B and C given by

$$e_a = e_m \sin\omega t; \quad e_b = e_m \sin(\omega t - 2\pi/3); \quad e_c = e_m \sin(\omega t - 4\pi/3) \Leftarrow$$

The instantaneous sum of the phase e.m.f.s (and also the phasor sum of the corresponding r.m.s. phasors E_a, E_b and E_c) is zero.

Asymmetric three-phase system The asymmetric system in *Figure 3.21*(*b*) has, in general, unequal phase voltages and phase displacements. Such asymmetry may occur in machines with unbalanced phase windings and in power supply systems when faults occur; the usual method of dealing with asymmetry is described in Section 3.2.12. Attention here is confined to the basic symmetric cases.

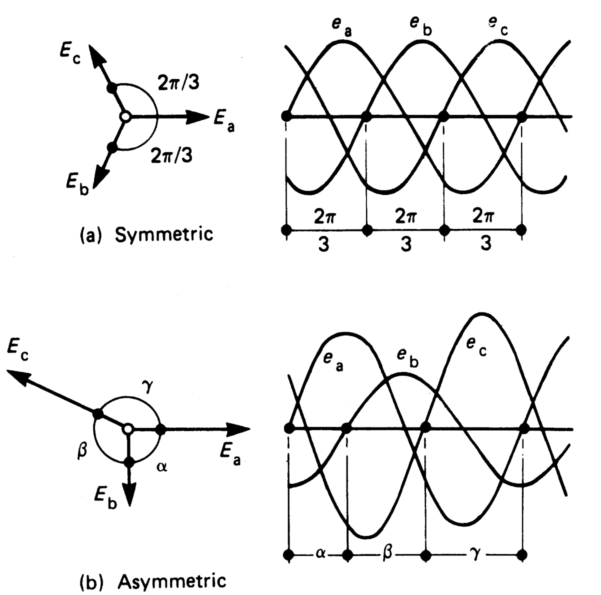


Figure 3.21 Three-phase systems

3.2.11.1 Phase interlinkage

While individual phase sources can be used separately, they are generated in the same machine and are normally interlinked. Using r.m.s. phasors, let the e.m.f. in a generator winding XY be such as to drive positive current out at X: then X is positive to Y and its e.m.f. E_{XY} is represented by an arrow with its point at X. The e.m.f. E_{YX} between terminals Y and X is therefore $-E_{XY}$. Further, when two windings XN and YN have a common terminal N, the e.m.f.s are

X to Y: $E_{XY} = E_{XN} - E_{YN}$

Y to X: $E_{YX} = E_{YN} - E_{XN} = -E_{XY}$

Common phase interconnections are shown in *Figure 3.22*.

3.2.11.2 Star

Let the phase e.m.f.s be E_{an}, E_{bn} and E_{cn} with an arbitrary positive direction outward from the star-point N. Then the line e.m.f.s are

$$E_{ab} = E_{an} - E_{bn}; \quad E_{bc} = E_{bn} - E_{cn}; \quad E_{ca} = E_{cn} - E_{an}$$

These are of magnitude $\sqrt{3}$ times that of a phase e.m.f., and provide a symmetric three-phase system of line e.m.f.s, with E_{ab} leading E_{an} by 30°. Thus $E_l = \sqrt{3}\,E_{ph}$ and $I_l = I_{ph}$, the subscripts l and ph referring to line and phase quantities respectively.

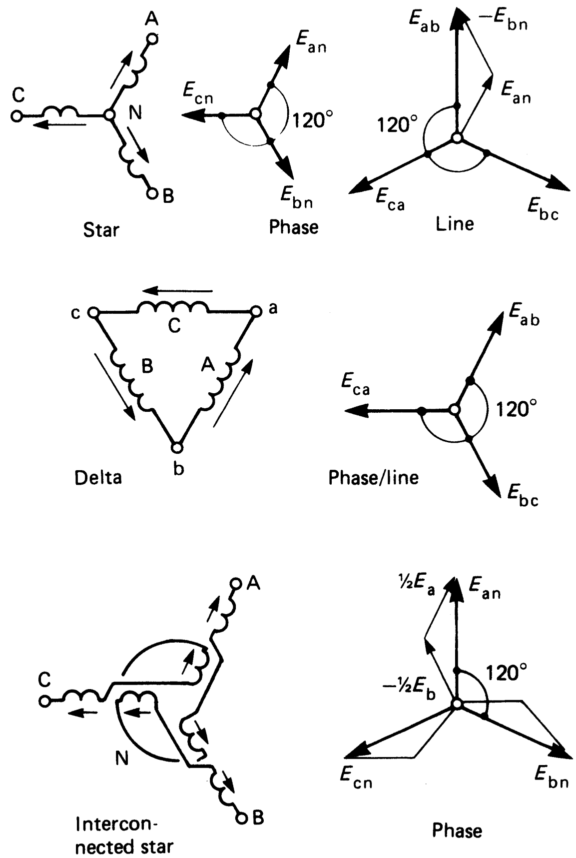


Figure 3.22 Three-phase interconnections

3.2.11.3 Delta

The line-to-line e.m.f. is that of the phase across which the lines are connected. The line current is the difference of the currents in the phases forming the line junction, so that the relations for symmetric loading are $E_l = E_{ph}$ and $I_l = \sqrt{3}\,I_{ph}$.

3.2.11.4 Interconnected star

A line-to-neutral e.m.f. comprises contributions from successive half-phases and sums to $\frac{1}{2}\sqrt{3}$ of a complete phase e.m.f. The line-to-line e.m.f. is $1\frac{1}{2}$ times the magnitude of a complete phase e.m.f. and the line current is numerically equal to the phase current.

3.2.11.5 Power

The total power delivered to or absorbed by a polyphase system, be it symmetric and balanced or not, is the algebraic sum of the individual phase powers. Consider an m-phase system with instantaneous line currents $i_1, i_2, \ldots, i_m$, the algebraic sum of which is zero by the Kirchhoff node law. Let the voltages of the input (or output) terminals, with reference to a common point X, be $v_1 - v_x$, $v_2 - v_X, \ldots, v_m - v_x$; then the instantaneous powers will be $(v_1 - v_x)i_1$, $(v_2 - v_x)i_2, \ldots,$ $(v_m - v_x)i_m$, which together sum to the total instantaneous power p. There is no restriction on the choice of X; it can be any of the terminals, say M. In this case $v_m - v_x = v_m - v_m = 0$, and the power summation has only $m-1$ terms. The average power over a full period T is, therefore,

$$P = (1/T)\int_0^T [(v_1 - v_m)i_1 + \cdots + (v_{m-1} - v_m)i_{m-1}]\,\mathrm{d}t$$

The first term of the sum in brackets represents the indication of a wattmeter with i_1 in its current circuit and $v_1 - v_m$ across its volt circuit, i.e. connected between terminals 1 and M. It follows that three wattmeters can measure the power in a three-phase four-wire system, and two in a three-phase three-wire system. Some of the common cases are listed below.

(1) *Three-phase, four-wire, load unbalanced*—The connections are shown in *Figure 3.23*(*a*). Wattmeters W_1, W_2 and W_3 measure the phase powers separately. The total power is the sum of the indications:

$$P = P_1 + P_2 + P_3$$

(2) *Three-phase, four-wire, load balanced*—with the connections shown in *Figure 3.23*(*a*), all the meters read the same. Two of the wattmeters can be omitted and the reading of the remaining instrument multiplied by 3.

(3) *Three-phase, three-wire, load unbalanced*—two wattmeters are connected with their current circuits in any pair of lines, as in *Figure 3.23*(*b*). The total power is the algebraic sum of the readings, regardless of waveform. A two-element wattmeter summates the power automatically; with separate instruments, one will tend to read reversed under certain conditions, given below.

(4) *Three-phase, three-wire, load balanced*—with sinusoidal voltage and current the conditions in *Figure 3.23*(*c*) obtain. Wattmeters W_1 and W_2 indicate powers P_1 and P_2 where

$$P_1 = V_{ab}I_a \cos(30^\circ + \phi) = V_l I_l \cos(30^\circ + \phi)$$
$$P_2 = V_{cb}I_c \cos(30^\circ - \phi) = V_l I_l \cos(30^\circ - \phi)$$

The total active power $P = P_1 + P_2$ is therefore

$$P = V_l I_l[\cos(30^\circ + \phi) + \cos(30^\circ - \phi)] = \sqrt{3}V_l I_l \cos\phi$$

where $\cos\phi$ is the *phase* power factor. The algebraic difference is $P_1 - P_2 = V_l I_l \sin\phi$, whence the reactive power is given by

$$Q = \sqrt{3}V_l I_l \sin\phi = \sqrt{3}(P_1 - P_2)$$

and the phase angle can be obtained from $\phi = \arctan(Q/P)$. For $\phi = 0$ (unity power factor) both wattmeters read alike; for $\phi = 60^\circ$ (power factor 0.5 lag) W_1 reads zero; and for lower lagging power factors W_1 tends to read backwards.

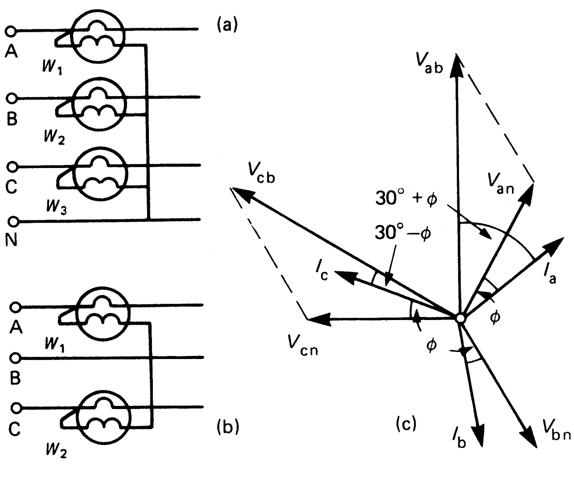


Figure 3.23 Three-phase power measurement

The active power of a single phase has a double-frequency pulsation (*Figure 3.18*). For the asymmetric two-phase system under balanced conditions and a phase displacement of 90°, and for all symmetric systems with $m = 3$ or more, the total power is constant.

3.2.11.6 Harmonics

Considering a symmetrical balanced system of three-phase non-sinusoidal voltages, and omitting phase displacements (which are in the context not significant), let the voltage of phase A be

$$v_a = v_1 \sin\,\omega t + v_2 \sin\,2\omega t + v_3 \sin\,3\omega t + \cdots$$

Writing $\omega t - \frac{2}{3}\pi$ and $\omega t - \frac{4}{3}\pi$, respectively, for phases B and C, and simplifying, we obtain

$$v_a = v_1 \sin\,\omega t + v_2 \sin\,2\omega t + v_3 \sin\,3\omega t + \cdots$$
$$v_b = v_1 \sin\,(\omega t - \tfrac{2}{3}\pi) + v_2 \sin\,2(\omega t - \tfrac{4}{3}\pi) + v_3 \sin\,3\omega t + \cdots$$
$$v_c = v_1 \sin\,(\omega t - \tfrac{4}{3}\pi) + v_2 \sin\,2(\omega t - \tfrac{2}{3}\pi) + v_3 \sin\,3\omega t + \cdots$$

The fundamentals have a normal $2\pi/3$ rad (120°) phase relation in the sequence ABC, as also do the 4th, 7th, 10th, ..., harmonics. The 2nd (and 5th, 8th, 11th, ...) harmonics have the $2\pi/3$ rad phase relation but of reversed sequence ACB. The *triplen* harmonics (those of the order of a multiple of 3) are, however, *co-phasal* and form a zero-sequence set.

The relation $V_l = \sqrt{3}\,V_{ph}$ in a three-phase star-connected system is applicable only for sine waveforms. If harmonics are present, the line- and phase-voltage waveforms differ because of the effective phase angle and sequence of the harmonic components. The nth harmonic voltages to neutral in two successive phases AB are $v_n \sin n\omega t$ and $v_n \sin n(\omega t - \frac{2}{3}\pi)$, and between the corresponding line terminals the nth harmonic voltage is $2v_n \sin n(\frac{1}{3}\pi)$. For triplen harmonics this is zero; hence no triplens are present in balanced line voltages because, in the associated phases, their components are equal and in opposition. In a balanced delta connection, again no triplens are present between lines: the delta forms a closed circuit to triplen circulating currents, the impedance drop of which absorbs the harmonic e.m.f.s.

3.2.12 Symmetrical components

Figure 3.21(a) shows the sine waves and phasors of a balanced symmetric three-phase system of e.m.f.s of sequence ABC. The magnitudes are equal and the phase displacements are $2\pi/3$ rad. In *Figure 3.21(b)*, the asymmetric sine waveforms have also the sequence ABC, but they are of different magnitudes and have the phase displacements α, β and γ. Problems of asymmetry occur in the unbalanced loading of a.c. machines and in fault conditions on power networks. While a solution is possible by the Kirchhoff laws, the method of *symmetrical components* greatly simplifies analysis.

Any set of asymmetric three-phase e.m.f.s or currents can be resolved into a summation of three sets of symmetrical components, respectively of positive phase-sequence (p.p.s.) ABC, negative phase-sequence (n.p.s.) ACB, and zero phase-sequence (z.p.s.). Use is made of the operator α, resembling the 90° operator j (Section 3.2.9.1) but implying a counter-clockwise rotation of $2\pi/3$ rad (120°). Thus

$$\alpha = 1 \angle 120° = \tfrac{1}{2}(-1 + \mathrm{j}\sqrt{3})$$
$$\alpha^2 = 1 \angle 240° = \tfrac{1}{2}(-1 - \mathrm{j}\sqrt{3})$$
$$\alpha^3 = 1 \angle 360° = 1 + \mathrm{j}0$$
$$1 + \alpha + \alpha^2 = 0$$

A symmetric three-phase system has only p.p.s. components

$$E_a = E_{a+}; \quad E_b = \alpha^2 E_{a+}; \quad E_c = \alpha E_{a+}$$

whereas an asymmetric system (*Figure 3.24*) comprises the three sets

$$\text{z.p.s.}\ E_{a0}; \quad E_{b0} = E_{a0}; \quad E_{c0} = E_{a0}$$
$$\text{p.p.s.}\ E_{a+}; \quad E_{b+} = \alpha^2 E_{a+}; \quad E_{c+} = \alpha E_{a+}$$
$$\text{n.p.s.}\ E_{a-}; \quad E_{b-} = \alpha E_{a-}; \quad E_{c-} = \alpha^2 E_{a-}$$

where the subscripts 0, + and − designate the z.p.s., p.p.s. and n.p.s. components, respectively. The p.p.s. and the n.p.s. components sum individually to zero. Therefore, if the originating phasors E_a, E_b, E_c also sum to zero there are no z.p.s. components; if they do not, their residual is the sum of the three z.p.s. components.

The asymmetrical phasors have now been reduced to the sum of three sets of symmetrical components:

$$E_a = E_{a0} + E_{a+} + E_{a-}$$
$$E_b = E_{b0} + E_{b+} + E_{b-}$$
$$E_c = E_{c0} + E_{c+} + E_{c-}$$

The components are evaluated from the arbitrary identities

$$E_a = Z + P + N$$
$$E_b = Z + \alpha^2 P + \alpha N$$
$$E_c = Z + \alpha P + \alpha^2 N$$

where

$$Z = (E_a + E_b + E_c)/3$$

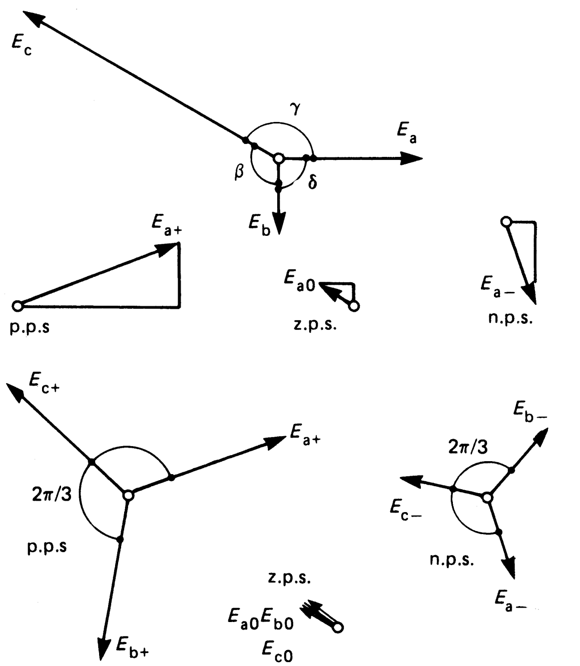


Figure 3.24 Symmetrical components

$$P = (E_a + \alpha E_b + \alpha^2 E_c)/3$$
$$N = (E_a + \alpha^2 E_b + \alpha E_c)/3$$

Figure 3.24 is drawn for an asymmetric system with voltages $E_a = 200$, $E_b = 100$ and $E_c = 400$ V, and phase-displacement angles $\delta = 90°$, $\beta = 120°$ and $\gamma = 150°$. In phasor terms,

$$E_a = 200 \angle 0° = 200 + \mathrm{j}0\ \mathrm{V}$$
$$E_b = 100 \angle (-90°) = 0 - \mathrm{j}100\ \mathrm{V}$$
$$E_c = 400 \angle 150° = -346 + \mathrm{j}200\ \mathrm{V}$$

Then

$$Z = (200 - \mathrm{j}100 - 346 + \mathrm{j}200)/3 = -49 + \mathrm{j}33 = E_{a0}$$
$$P = (200 + 87 + \mathrm{j}50 + 347 + \mathrm{j}200)/3 = 211 + \mathrm{j}83 = E_{a+}$$
$$N = (200 - 87 + \mathrm{j}50 - \mathrm{j}400)/3 = 38 - \mathrm{j}117 = E_{a-}$$

The summation $E_{a0} + E_{a+} + E_{a-} = 200 + \mathrm{j}0 = E_a$. The p.p.s and n.p.s. components of E_b and E_c are readily obtained.

3.2.12.1 Power

In linear networks there is no interference between currents of different sequences. Thus p.p.s. voltages produce only p.p.s. currents, etc. The total power is therefore

$$P = P_a + P_b + P_c$$
$$= 3(V_0 I_0 \cos\phi_0 + V_+ I_+ \cos\phi_+ + V_- I_- \cos\phi_-)$$

This is equivalent to the more obvious summation of phase powers

$$P = V_a I_a \cos\phi_a + V_b I_b \cos\phi_b + V_c I_c \cos\phi_c$$

Symmetrical-component techniques are useful in the analysis of power-system networks with faults or unbalanced loads: an example is given in Section 3.3.4. Machine performance is also affected when the machine is supplied from an asymmetric voltage system: thus in a three-phase induction motor the n.p.s. components set up a torque in opposition to that of the (normal) p.p.s. voltages.

3.2.13 Line transmission

Networks of small physical dimensions and operated at low frequency are usually considered to have a zero propagation time; a current started in a closed circuit appears at every point in the circuit simultaneously. In extended circuits, such as long transmission lines, the propagation time is significant and cannot properly be ignored.

The basics of energy propagation on an ideal loss-free line are discussed in another section. Propagation takes place as a voltage wave v accompanied by a current wave i such that $v/i = z_0$ (the surge impedance) travelling at speed u. Both z_0 and u are functions of the line configuration, the electric and magnetic space constants ϵ_0 and μ_0, and the relative permittivity and permeability of the medium surrounding the line conductors. At the receiving end of a line of finite length, an abrupt change of the electromagnetic-field pattern (and therefore of the ratio v/i) is imposed by the discontinuity unless the receiving-end load is z_0, a termination called the *natural load* in a power line and a *matching impedance* in a telecommunication line. For a non-matching termination, wave reflection takes place with an electromagnetic wave running back towards the sending end. After many successive reflections of rapidly diminishing amplitude, the system settles down to a steady state determined by the sending-end voltage, the receiving-end load impedance and the line parameters.

3.2.13.1 A.c. power transmission

The steady-state condition considered is the transfer of a constant balanced apparent power per phase from a generator at the sending end (s) to a load at the receiving end (r) by a sinusoidal voltage and current at a frequency $f = \omega/2\pi$. The line has uniformly distributed parameters: a conductor resistance r and a loop inductance L effectively in series, and an insulation conductance g and capacitance C in shunt, all per phase and per unit length. The series impedance, shunt admittance and propagation coefficient per unit length are $z = r + j\omega L$, $y = g + j\omega C$ and $\gamma = \sqrt{(yz)}$, respectively. For a line of length l the overall parameters are $zl = Z$, $yl = Y$ and $l\sqrt{(yz)} = \sqrt{(YZ)} = \gamma l$. The solution for the receiving-end terminal conditions is in terms of $\sqrt{(YZ)}$ and its hyperbolic functions as a two-port:

$$V_s = V_r\boldsymbol{A} + I_r\boldsymbol{B} = V_r\cosh(\sqrt{YZ}) + I_r z_0 \sinh(\sqrt{YZ})$$

$$I_s = V_r\boldsymbol{C} + I_r\boldsymbol{D} = V_r(1/z_0)\sinh(\sqrt{YZ}) + I_r\cosh(\sqrt{YZ})$$

Using the hyperbolic series (Section 1.2.2) and writing $z_0 = \sqrt{(Z/Y)}$, we obtain for a symmetrical line

$$\boldsymbol{A} = 1 + YZ/2 + (YZ)^2/24 + \cdots = \boldsymbol{D}$$

$$\boldsymbol{B} = Z[1 + YZ/6 + (YZ)^2/120 + \cdots]$$

$$\boldsymbol{C} = Y[1 + YZ/6 + (YZ)^2/120 + \cdots]$$

The significance of the higher powers of YZ depends on: (i) the line configuration, (ii) the properties of the ambient medium, and (iii) the physical length of the line in terms of the wavelength $\lambda = u/f$. For air-insulated overhead lines the inductance is large and the capacitance small: the propagation velocity approximates to $u = 3\times 10^5$ km/s (corresponding to a wavelength $\lambda = 6000$ km at 50 Hz), with a natural load z_0 of the order of 400–500 Ω. Cable lines have a low inductance and a large capacitance: the permittivity of the dielectric material and the presence of armouring and sheathing result in a propagation velocity around 200 km/s, a surge impedance below 100 Ω, and the possibility (in high-voltage cables) that the charging current may be comparable with the load current if the cable length exceeds 25–30 km.

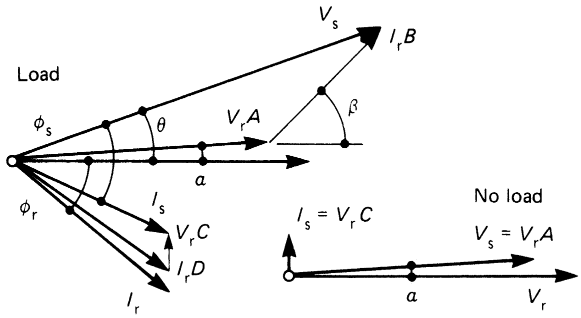


Figure 3.25 Transmission-line phasor diagram

For balanced three-phase power transmission, the general equations are applied for the line-to-neutral voltage, line current and phase power factor. Phasor diagrams for the load and no-load ($I_r = 0$) receiving-end conditions for an overhead-line transmission are shown in *Figure 3.25*, with V_r as datum. On no load, $V_s = V_r\boldsymbol{A}$, and as $\boldsymbol{A}$ has a magnitude less than unity and a small positive angle α, the phasor $V_r\boldsymbol{A}$ is smaller than V_r and leads it by angle α: thus $V_r > V_s$, the *Ferranti effect*. For the loaded condition, $I_r\boldsymbol{B}$ is added to $V_r\boldsymbol{A}$ to give V_s. Similarly $V_r\boldsymbol{C}$ is added to $I_r\boldsymbol{D}$ to obtain I_s.

The product $V_r I_r = I_r(V_s - V_r A)$ is the receiving-end complex apparent power S_r. Let V_s lead V_r by angle θ; then the receiving-end load has the active and reactive powers P_r and Q_r given by

$$P_r = (V_s V_r/\boldsymbol{B})\cos(\theta - \beta) - (V_r^2\boldsymbol{A}/\boldsymbol{B})\cos(\beta - \alpha)$$

$$Q_r(V_s V_r/\boldsymbol{B})\sin(\theta - \beta) + (V_r^2\boldsymbol{A}/\boldsymbol{B})\sin(\beta - \alpha)$$

where α and β are the angles in the complexors $\boldsymbol{A}$ and $\boldsymbol{B}$. The importance of $\boldsymbol{B}$ (roughly the overall series impedance) is clear.

Line chart Operating charts for a transmission circuit can be drawn to relate graphically V_s, V_r, P_r and Q_r, using the appropriate overall ***ABCD*** parameters (e.g. with terminal transformers included).

Receiving-end chart A receiving-end chart gives active and reactive power at the receiving end for V_r constant (*Figure* 3.26(*a*)). The co-ordinates (x, y) and the radius (r) of the constant-voltage circles are

$$x = -V_r^2(\boldsymbol{A}/\boldsymbol{B})\cos(\beta - \alpha)$$

$$y = -V_r^2(\boldsymbol{A}/\boldsymbol{B})\sin(\beta - \alpha)$$

$$r = V_s V_r/\boldsymbol{B}$$

where $\boldsymbol{A}$ and $\boldsymbol{B}$ are scalar magnitudes, and α and β the angles in $\boldsymbol{A}$ and $\boldsymbol{B}$. For a given V_r the chart comprises a family of concentric circles, each corresponding to a particular V_s. If a given receiving-end load is located by its active and reactive power components, V_s is obtained from the corresponding V_s circle.

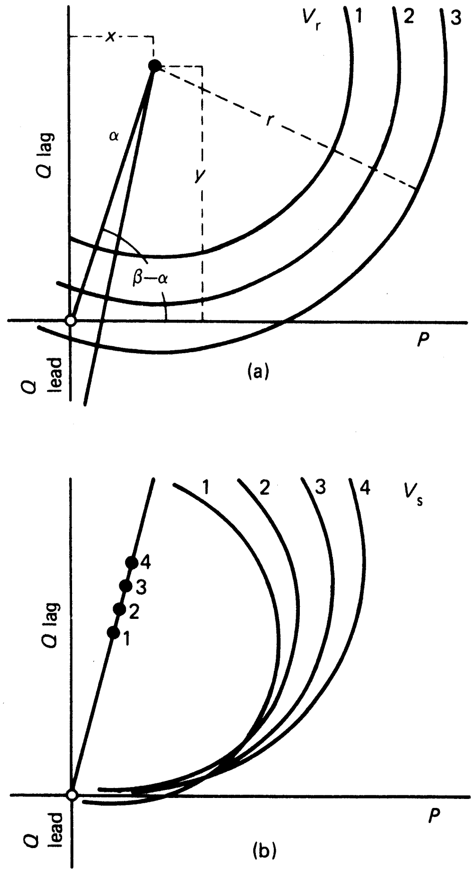


Figure 3.26 Line charts

Sending-end chart For a given V_s the sending-end chart comprises a family of circles as shown in *Figure 3.26(b)*, each circle corresponding to a particular V_r. Load points outside the envelope of these circles cannot be supplied at the V_s for which the chart is drawn.

3.2.13.2 Short line

For an overhead interconnector line the capacitive shunt admittance is neglected, reducing the general parameters to $\boldsymbol{A} = \boldsymbol{D} = 1$, $\boldsymbol{B} = Z = \boldsymbol{R} + \mathrm{j}X$ and $\boldsymbol{C} = 0$. The operating conditions are those in *Figure 3.27*, with a receiving-end voltage V_r (taken as reference phasor), a sending-end voltage V_s and a load current I at a lagging phase angle ϕ with respect to V_r and having active and reactive components respectively I_p and I_q. Then

$$V_s = V_r + (I_p - \mathrm{j}I_q)(R + \mathrm{j}X) \Leftarrow$$
$$= V_r + (I_pR + I_qX) + \mathrm{j}(I_pX - I_qR) = V_r + v + \mathrm{j}u$$

To a close approximation, v is the difference of the voltages V_s and V_r, while u determines their phase difference (or transmission angle). The regulation and angle are therefore v/V_s p.u. and $\theta = \arctan(u/V_s)$ rad.

Suppose that $V_r = V_s$; then $v = 0$ giving $I_q = -I_p(R/X)$, and $u = I_pX[1 + (R/X)^2]$ giving $\theta = \arctan(I_p/V_s)[1 + (R/X)^2]$. The consequences are that (i) for a receiving-end active power P the load must be able to absorb a leading

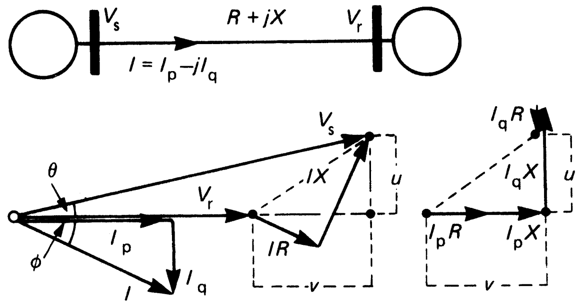


Figure 3.27 Operating conditions for a short transmission line

reactive power $Q = P(R/X)$, and (ii) the transmission angle is determined largely by X. If $R/X = 0.5$, typical of an overhead line, then $Q = 0.5P$ and $\theta = \arctan[1.25\ X(I_p/V_s)]$. With interconnector cables the R/X ratio is usually greater than unity and shunt capacitance current is no longer negligible.

Independent adjustment of V_s and V_r is not feasible, and effective load control requires adjustment of v (e.g. by transformer taps) and of u (e.g. by quadrature boosting).

3.2.14 Network transients

Energy cannot be instantaneously converted from one form to another, although the time needed for conversion can be very short and the conversion rate (i.e. the power) high. Change between states occurs in a period of *transience* during which the system energies are redistributed in accordance with the energy-conservation principle (Section 1.3.1). For example, in a simple series circuit of resistance R, inductance L and capacitance C connected to a source of instantaneous voltage v, the corresponding rates of energy input, dissipation (in R) and storage (in L and C) are related by

$$p = vi = Rii + [L(\mathrm{d}i/\mathrm{d}t)i + (q/C)i] \Leftarrow$$

Dividing by the common current i and writing the capacitor charge q as the time-integral of the current gives the voltage equation

$$v = Ri + L(\mathrm{d}i/\mathrm{d}t) + (1/C)\int i\ \mathrm{d}t$$

and any changes in the parameters or in the applied voltage demand changes in the distribution of the circuit energy. The integro-differential equation can be solved to yield both steady-state and transient conditions.

In practical circuits the system may be too complex for such a direct solution; the following methods may then be attempted:

(1) formal mathematics for simple cases with linear parameters;
(2) simplification, e.g. by linearising parameters or by neglecting second-order terms;
(3) writing a possible solution based on the known physical behaviour of the system, with a check by differentiation;
(4) setting up a model system on an analogue computer; or
(5) programming a digital computer to give a solution by iteration.

3.2.14.1 Classification

Where the system has only one energy-storage component, *single-energy* transients occur. Where two (or more) different storages are concerned, the transient has a *double-* (or *multiple-*) *energy* form. Transients may occur in the following circumstances.

(1) *Initiation*—a system, initially dead, is energised.
(2) *Subsidence*—an initially energised system is reduced to a zero-energy condition.
(3) *Transition*—a change from one state to another, where both states are energetic.
(4) *Complex*—the superposition of more than one disturbance.
(5) *Relaxation*—transition between states that, when reached, are themselves unstable.

Further distinctions can be made, e.g. between linear and non-linear parameters, neglect or otherwise of propagation time within the system, etc. Attention here is mainly confined to simple electric networks with constant parameters and, by analogy (Section 1.3.1), to corresponding mechanical systems.

3.2.14.2 Transient forms

During transience, the current i for an impressed voltage stimulus $v(t)$ is considered to be the superposition of a transient component i_t and a final steady-state current i_s, so that at any instant $i = i_s + i_t$. Alternatively, the voltage v for an impressed current stimulus $i(t)$ is the summation $v = v_s + v_t$. The quantities i_s or v_s are readily derived by applying the appropriate steady-state technique. The *form* of i_t or v_t is characteristic of the system itself, is independent of the stimulus and comprises exponential terms $k\exp(\lambda t)$ where k depends on the boundary conditions. This is the case because of the fixed proportionality between the stored energy $\frac{1}{2}Li^2$ and the rate of energy dissipation Ri^2 in an RL circuit; and similarly for $\frac{1}{2}Cv^2$ and v^2/R in an RC circuit. Hence the transient form can be obtained from a case in which the final steady state is of zero energy, i.e. a subsidence transient.

The subsidence transient in a *single-energy* (first-order) system having the general equation $dy/dt + ay = 0$ can be found by substituting λ for d/dt to give $\lambda y + ay = 0$, whence $\lambda = -a$. Then the solution is

$$y = k\exp(\lambda t) = k\exp(-at)$$

a simple exponential decay as in *Figure 1.2* of Section 1.2.2. For a *double-energy* (second-order) system the basic equation is

$$d^2y/dt^2 + a(dy/dt) + by = 0 \quad \text{or} \quad \lambda^2 + a\lambda + b = 0$$

The quadratic in λ has two roots, λ_1 and λ_2, and the solution has a pair of exponential terms that depend on the relation between a and b. For a *multiple-energy* (nth-order) system there will be n roots. From Section 1.2.2 it will be seen that exponential terms can represent oscillatory as well as decay forms of response.

Single-energy system Consider the RL circuit shown in *Figure 3.28*, subsequent to closure of the switch at $t = 0$. The transient current form is obtained from $L(di/dt) + Ri = 0$, or $L\lambda i + Ri = 0$, giving $\lambda = -R/L$. Then

$$i_t = k\exp[-t(R/L)] = k\exp(-t/T)$$

where $T = L/R$ is the *time-constant*. The final steady-state current depends on the source voltage v. In *Figure 3.28(a)*, with $v = V$, a constant direct voltage, $i_s = V/R$. Immediately after switching, with $t = 0+$, the current i is still zero because the inductance prevents any instantaneous rise. Hence

$$i = i_s + i_t = V/R + k\exp(-0) = V/R + k$$

so that $k = -(V/R)$. From $t = 0$ the current is, therefore,

$$i = i_s + i_t = (V/R)[1 - \exp(-t/T)]$$

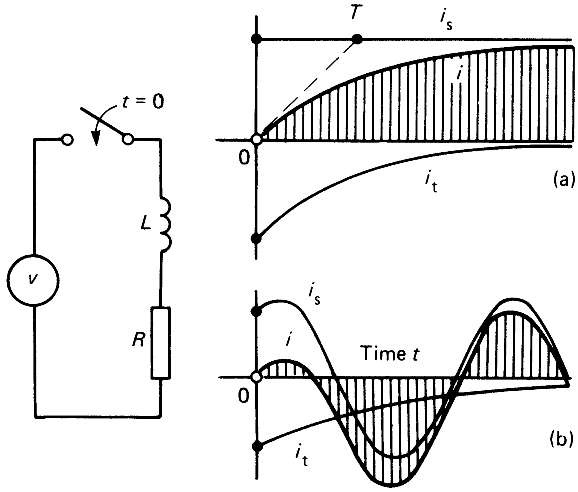


Figure 3.28 Transients in an inductive–resistive circuit

The two terms and their summation are shown in *Figure 3.28(a)*.

If, as in *Figure 3.28(b)*, the source voltage is sinusoidal expressed by $v = v_m \sin(\omega t - \alpha)$ and again switching occurs at $t = 0$, the form of the transient current is unchanged, but the final steady-state current is

$$i_s = (v_m/Z)\sin(\omega t - \alpha - \phi)$$

where $Z = \sqrt{(R^2 + \omega^2L^2)}$ and $\phi = \arctan(\omega L/R)$. At $t = 0+$

$$i = i_s + i_t = (v_m/Z)\sin(-\alpha - \phi) + k = 0$$

which gives $k = -(v_m/A)\sin(-\alpha - \phi)$. The final steady-state and transient current components are shown in *Figure 3.28(b)* with their resultant. Initially the current is asymmetric, but subsequently the decay of i_t allows the current to approach the steady-state condition.

If $\omega L \gg R$, then approximately $\phi = \frac{1}{2}\pi$. Let the switch be closed at $v = 0$ for which $\alpha = 0$. Then the current is

$$i = (v_m/\omega L)[\sin(\omega t - \tfrac{1}{2}\pi) + 1]$$

which raises i to twice the normal steady-state peak when t reaches a half-period: this is the *doubling effect*. However, if the switch is closed at a source-voltage maximum, the current assumes its steady-state value immediately, with no transient component.

Summary for an RL circuit The transient current has a decaying exponential form, with a value of k such that, when it is added to the final steady-state current, the initial current flowing in the circuit at $t = 0$ is obtained. (In both of the cases in *Figure 3.28* the initial current is zero.) Thus if the initial circuit current is 10 A and the final current is 25 A, the value of k is −15 A.

For the CR circuit in *Figure 3.29*, the form of the transient is found from $Ri + q/C = 0$; differentiating, we obtain

$$R(di/dt) + (1/C)i = 0 \quad \text{or} \quad R\lambda + 1/C = 0$$

from which $\lambda = -1/CR = -1/T$, where $T = CR$ is the *time-constant*. Thus $i_t = k\exp(-t/T)$. With the capacitor initially uncharged and a source direct voltage V switched on at $t = 0$,

$$i = i_s + i_t = 0 + k\exp(-t/T)$$

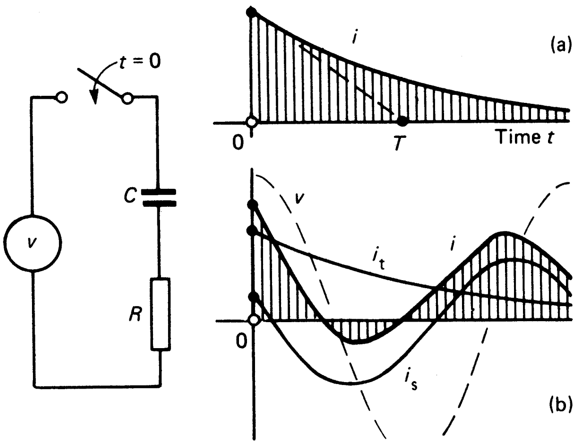


Figure 3.29 Transients in a capacitive–resistive circuit

As this must be V/R at $t=0+$, then $k=V/R$, as shown in *Figure 3.29(a)*. In *Figure 3.29(b)* the initiation of a *CR* circuit with a sine voltage is shown.

Summary for an RC circuit The transient current is a decaying exponential $k\ \exp(-t/T)$. The initial current is determined by the voltage difference between the voltage applied by the source and that of the capacitor. (In *Figure 3.29* the capacitor is in each case uncharged.) If this p.d. is V_0, then the initial current is V_0/R.

Double-energy system A typical case is that of a series *RLC* circuit. The transient form is obtained from $L(\mathrm{d}i/\mathrm{d}t) + Ri + q/C = 0$, differentiated to

$$\mathrm{d}^2i/\mathrm{d}t^2 + (R/L)(\mathrm{d}i/\mathrm{d}t) + (1/LC)i = 0$$

Thus $\lambda^2 + (R/L)\lambda + 1/LC = 0$ is the required equation, with the roots

$$\lambda_1, \lambda_2 = -\frac{R}{2L} \pm \left(\frac{R^2}{4L^2} - \frac{1}{LC}\right)^{1/2}$$

The resulting transient depends on the sign of the quantity in parentheses, i.e. on whether $R/2L$ is greater or less than $1/\sqrt{(LC)}$. Four waveforms are shown in *Figure 3.30*.

(1) *Roots real*: $R > 2\sqrt{(L/C)}$. The transient current is unidirectional and results from two simple exponential curves with different rates of decay.

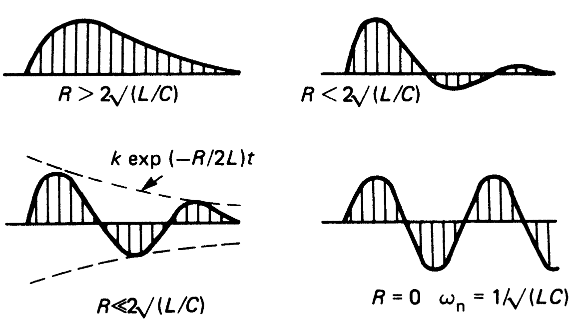


Figure 3.30 Double-energy transient forms

(2) *Roots equal*: $R = 2\sqrt{(L/C)}$. This has more mathematical than physical interest, but it marks the boundary between unidirectional and oscillatory transient current.

(3) *Roots complex*: $R < 2\sqrt{(L/C)}$. The roots take the form $-\alpha \pm \mathrm{j}\omega_n$, and the transient current oscillates with the interchange of magnetic and electric energies respectively in L and C; but the oscillation amplitude decays by reason of dissipation in R. With $R=0$ the oscillation persists without decay at the undamped natural frequency $\omega_n = 1/\sqrt{(LC)}$.

Pulse drive The response of networks to single pulses (or to trains of such pulses) is an important aspect of data transmission. An ideal pulse has a rectangular waveform of duration ('width') t_p. It can be considered as the resultant of two opposing step functions displaced in time by t_p as in *Figure 3.31(a)*.

In practice a pulse cannot rise and fall instantaneously, and often the amplitude is not constant (*Figure 3.31(b)*). Ambiguity in the precise position of the peak value V_p makes it necessary to define the *rise time* as the interval between the levels $0.1\ V_p$ and $0.9\ V_p$. The *tilt* is the difference between V_p and the value at the start of the trailing edge, expressed as a fraction of V_p.

The response of the output network to a voltage pulse depends on the network characteristics (in particular its time-constant T) and the pulse width t_p. Consider an ideal input

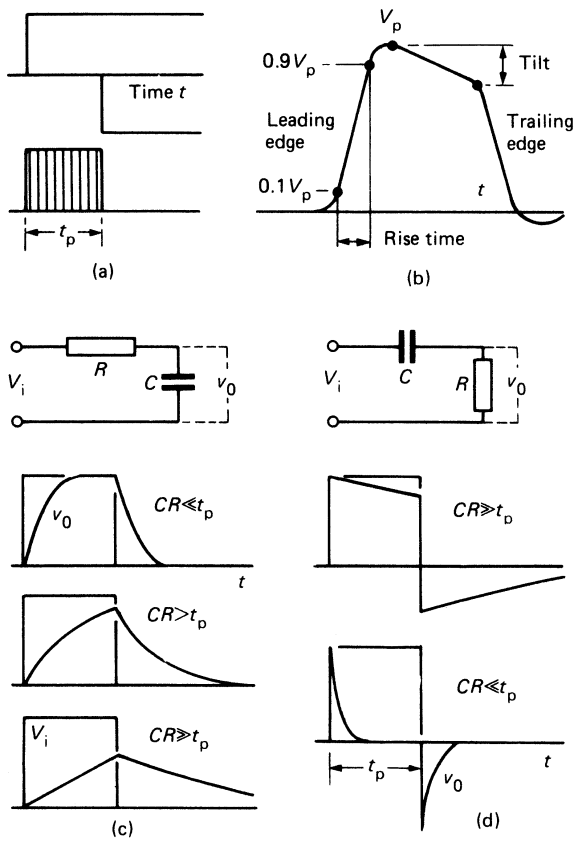


Figure 3.31 Pulse drive

voltage V_i of rectangular waveform applied to an ideal low-pass series network (*Figure 3.31(c)*), the output being the voltage v_0 across the capacitor C. Writing p for d/dt, then

$$\frac{v_0}{V_i} = \frac{1/\mathrm{p}C}{R + 1/\mathrm{p}C} = \frac{1}{1 + \mathrm{p}CR} = \frac{1}{1 + \mathrm{p}T}$$

where $T = CR$ is the network time-constant. This represents an exponential growth $v_0 = V_i[1 - \exp(-t/T)]$ over the interval t_p. The trailing edge is an exponential decay, with t reckoned from the start of the trailing edge. Three typical responses are shown. For $CR \ll t_p$ the output voltage reaches V_i; for $CR \simeq t_p$ the rise is slow and does not reach V_i; for $CR \gg t_p$ the rise is almost linear, the final value is small and the response is a measure of the time-integral of V_i.

With C and R interchanged as in *Figure 3.31(d)* to give a high-pass network, the whole of V_i appears across R at the leading edge, falling as C charges. Following the input pulse there is a reversed v_0 during the discharge of the capacitor. The output/input voltage relation is given by

$$\frac{v_0}{V_i} = \frac{R}{R + 1/\mathrm{p}C} = \frac{\mathrm{p}CR}{1 + \mathrm{p}CR} = \frac{\mathrm{p}T}{1 + \mathrm{p}T}$$

For $CR \gg t_p$ the response shows a tilt; for $CR \ll t_p$ the capacitor charges rapidly and the output v_0 comprises positive- and negative-going spikes that give a measure of the time-differential of V_i.

3.2.14.3 Laplace transform method

Application of the Laplace transforms is the most usual method of solving transient problems. The basic features of the Laplace transform are set out in Section 1.2.7 and *Table 3.4*, which gives transform pairs. The advantages of the method are that: (1) any stimulus, including discontinuous and pulse forms, can be handled, (2) the solution is complete with both steady-state and transient components, (3) the initial conditions are introduced at the start, and (4) formal mathematical processes are avoided.

Consider the system in *Figure 3.28(a)*. The applied direct voltage V has the Laplace transform $V(s) = V/s$; the operational impedance of the circuit is $Z(s) = R + Ls$. Then the Laplace transform of the current is

$$I(s) = \frac{V(s)}{Z(s)} = \frac{V}{s}\frac{1}{R + Ls} = \frac{V}{L}\frac{1}{s(s + R/L)}$$

The term V/L is a constant unaffected by transformation. The term in s is almost of the form $a/s(s + a)$. So, if we write

$$I(s) = \frac{V}{aL}\frac{a}{s(s + a)}$$

where $a = R/L = 1/T$, the inverse Laplace transform gives

$$i(t) = (V/aL)[1 - \exp(-at)] = (V/R)[1 - \exp(-t/T)]$$

which is the complete solution. More complex problems require the development of partial fractions to derive recognisable transforms which are then individually inverse-transformed to give the terms in the solution of $i(t)$.

3.2.15 System functions

It is characteristic of linear constant-coefficient systems that their operational solution involves three parts: (i) the excitation or stimulus, (ii) the output or response and (iii) the system function. Thus in the relation $I(s) = V(s)/Z(s)$ for the current in Z resulting from the application of V, $1/Z(s)$ is the *system function* relating voltage to current. For the simple electrical system shown in *Figure 3.32*(*a*) the system function $Y(s)$ relating $V(s)$ to $I(s)$ in $I(s) = V(s)Y(s)$ is $Y(s) = 1/(R + Ls + 1/Cs)$. Different functions could relate the capacitor charge or the magnetic linkage in the inductor to the transform $V(s)$ of the stimulus $v(t)$.

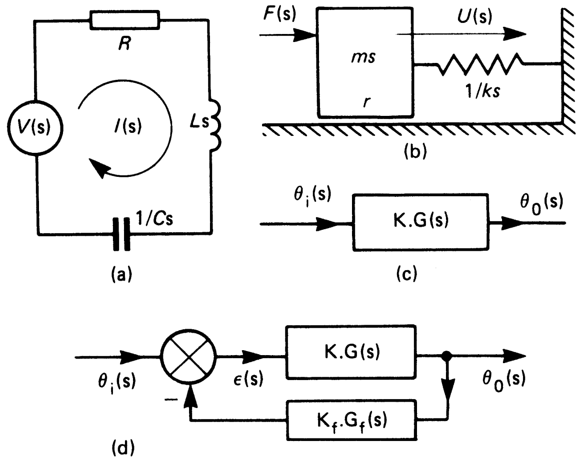


Figure 3.32 System functions

The mechanical analogue (*Figure 3.32(b)*) of this electrical system, as indicated in Section 1.3.1, has a system transfer function to relate force $f(t)$ to velocity $u(t)$ of the mass m and one end of the spring of compliance k in the presence of viscous friction of coefficient r. Then $F(s)$ and $U(s)$ are the transforms of $f(t)$ and $u(t)$, and the operational 'mechanical impedance' has the terms ms, $1/ks$ and r. In general, an input $\theta_i(s)$ and an output $\theta_o(s)$ are related by a system transfer function $KG(s)$ (*Figure 3.32(c)*), where K is a numerical or a dimensional quantity to include amplification or the value of some physical quantity (such as admittance). The transform of the integro-differential equation of variation with time is expressed by the term $G(s)$. The system is then represented by the block diagram in *Figure 3.32*(*c*); i.e. $\theta_o(s)/\theta_i(s) = KG(s)$.

A number of typical system transfer functions for relatively simple systems are given in *Table 3.4*.

The output of one system may be used as the input to another. Provided that the two do not interact (i.e. the individual transfer functions are not modified by the connection) the overall system function is the product $[K_1G_1(s)] \times [K_2G_2(s)]$ of the individual functions. If the systems are paralleled and their outputs are additively combined, the overall function is their sum.

3.2.15.1 Closed-loop systems

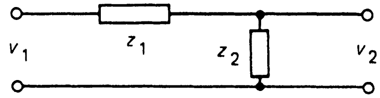
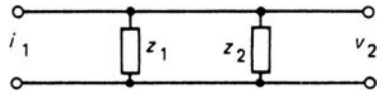
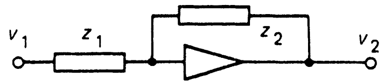
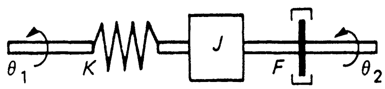
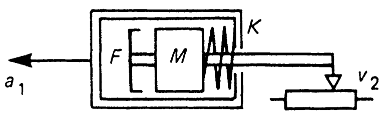
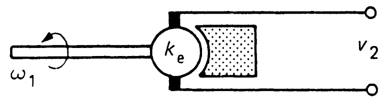
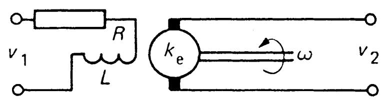
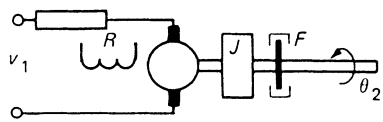
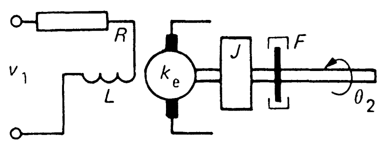
In *Figure 3.32*, parts (*a*), (*b*) and (*c*) are *open-loop* systems. However, the output can be made to modify the input by feedback through a network $K_fG_f(s)$ as in (*d*). The signal

$$\theta_f(s) = [K_fG_f(s)]\theta_o(s)$$

is combined with $\theta_i(s)$ to give the modified input.

For *positive feedback*, the resultant input is $\sigma(s) = \theta_i(s) + \theta_f(s)$, and the effect is usually to produce instability and oscillation.

Table 3.4 System transfer functions [the relation $f_2(t)/f_1(t)$ of output to input quantity in terms of the Laplace transform $F_2(s)/F_1(s)$]

System		*Transfer function*	
1 Electrical network		$\frac{V_2(s)}{V_1(s)} = \frac{Z_2(s)}{Z_1(s)+Z_2(s)}$	
2 Electrical network		$\frac{V_2(s)}{I_1(s)} = \frac{Z_1(s)Z_2(s)}{Z_1(s)+Z_2(s)}$	
3 Feedback amplifier		$\frac{V_2(s)}{V_1(s)} = \frac{Z_2(s)}{Z_1(s)}$	
4 Second-order system		$\frac{\theta_2(s)}{\theta_1(s)} = \frac{1}{1+2csT+s^2T^2}$	$T = \sqrt{(J/K)}$ $c = F/2\sqrt{(JK)}$
5 Accelerometer		$\frac{V_2(s)}{A_1(s)} = \frac{k_a}{1+2csT+s^2T^2}$	$T = \sqrt{(M/K)}$ $c = F/2\sqrt{(MK)}$
6 Permanent-magnet generator		$\frac{V_2(s)}{\omega_1(s)} = k_e$	
7 Separately excited generator		$\frac{V_2(s)}{V_1(s)} = \frac{k_e\omega}{R(1+sT)}$	$T = L/R$
8 Motor: armature control		$\frac{\theta_2(s)}{V_1(s)} = \frac{K_e}{s(1+sT)}$	$K_e = k_e/(FR+k_e^2)$ $T = JR/(FR+k_e^2)$
9 Motor: field control		$\frac{\theta_2(s)}{V_1(s)} = \frac{k_e}{s(1+sT_1)(1+sT_2)}$	$T_1 = J/F$, $T_2 = L/R$

v	voltage	L	inductance	θ	angular displacement	M	mass
i	current	k_e	e.m.f. coefficient	ω	angular velocity	J	inertia
Z	impedance	c	damping coefficient	a	acceleration	F	viscous friction coefficient
R, r	resistance	T	time-constant	k_a	acceleration coefficient	K	stiffness

For *negative feedback*, the resultant input is the difference $\epsilon(s) = \theta_i(s) - \theta_f(s)$, an 'error' signal. With the main system $KG(s)$ now relating ϵ and θ_o, the output/input relation is

$$\frac{\theta_o(s)}{\theta_i(s)} = \frac{KG(s)}{1+[KG(s)][K_fG_f(s)]}$$

Suppose that there is unity feedback $K_fG_f(s) = 1$; then if $KG(s)$ is large

$$\theta_o(s)/\theta_i(s) = KG(s)/[1+KG(s)] \approx 1$$

and the output closely follows the input in magnitude and wave shape, a condition sought in servo-mechanisms and feedback controls.

3.2.15.2 System performance

In general, a system function takes the form numerator/denominator, each a polynomial in s, relating response to input stimulus. Two forms are

$$KG(s) = \frac{b_m s^m + b_{m-1}s^{m-1} + \cdots + b_0}{a_n s^n + a_{n-1}s^{n-1} + \cdots + a_0} \quad (3.1)$$

$$= \frac{b_m(s-z_1)(s-z_2)\cdots(s-z_m)}{a_n(s-p_1)(s-p_2)\cdots(s-p_n)} \quad (3.2)$$

The response depends both on the system and on the stimulus. Performance can be studied if simple formalised stimuli (e.g. step, ramp or sinusoidal) are assumed; an exponential stimulus is even more direct because (in a linear system) the transient and steady-state responses are then both exponential. With the system function expressed in terms of the complex frequency $s = \sigma + j\omega$ it is necessary to express the stimulus in similar terms and to evaluate the response as a function of time by inverse Laplace transformation. The response in the *frequency domain* (i.e. the output/input relation for sustained sinusoidal stimuli over a frequency range) is obtained by taking $s = j\omega$ and solving the complexor $KG(j\omega)$. Another alternative is to derive the poles (p) and zeroes (z) in equation (3.2) above.

Thus there are several techniques for evaluating system functions. Some are graphical and give a concise representation of the response to specified stimuli.

3.2.15.3 Poles and zeros

In equation (3.2), the numbers z are the values of s for which $KG(s) = 0$; for, if s is set equal to z_1 or $z_2, \ldots$, the numerator has a zero term as a factor. Similarly, if s is set equal to p_1 or $p_2, \ldots$, there is a zero factor in the denominator and $KG(s)$ is infinite. Then the z terms are the *zeros* and the p terms are the *poles* of the system function. Except for the term b_m/a_n, the system function is completely specified by its poles and zeros.

Consider the network of *Figure 3.33*, the system function required being the output voltage v_o in terms of the input voltage v_i. This is the ratio of the paralleled branches R_2L_2C to the whole impedance across the input terminals. Algebra gives

$$KG(s) = 8(s+1)/(s^3 + 3s^2 + 14s + 16)$$

$$= 8\frac{s+1}{(s+1.36)[s+(0.82+j3.33)][s+(0.82-j3.33)]}$$

by factorising numerator and denominator. Thus there is one zero for $s = -1$. There are three poles, with $s = -1.36$, and $-0.82 \pm j3.33$. These are plotted on the complex s-plane in *Figure 3.33*. Poles on the real axis σ correspond to simple exponential variations with time, decaying for negative and increasing indefinitely for positive values. Poles in conjugate pairs on the $j\omega$ axis correspond to sustained sinusoidal oscillations. If the poles occur displaced from the origin and not on either axis, they refer to sinusoids with a decay or a growth factor, depending on whether the term σ is negative or positive.

3.2.15.4 Harmonic response

This is the steady-state response to a sinusoidal input at angular frequency ω. When a sine signal input is applied to a linear system, the steady-state response is also sinusoidal and is related to the input by a relative magnitude M and a phase angle α. The system function is $KG(j\omega)$.

Consider again the network of *Figure 3.33*. Writing $s = j\omega$ and simplifying gives the phasor expression for V_o/V_i as

$$KG(j\omega) = \frac{8(j\omega + 1)}{(16 - 3\omega^2) + j\omega(14 - \omega^2)} = |M| \angle \alpha$$

Plots of $|M|$ and $\angle \alpha$ are shown in *Figure 3.34(a)*. For $\omega = 0$ the network is a simple voltage divider with $V_o/V_i = 0.5$ and a phase angle $\alpha = 0$. For $\omega = \infty$, the terminal capacitor effectively

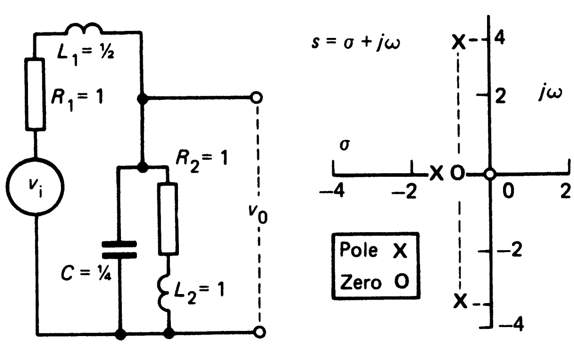


Figure 3.33 Poles and zeros

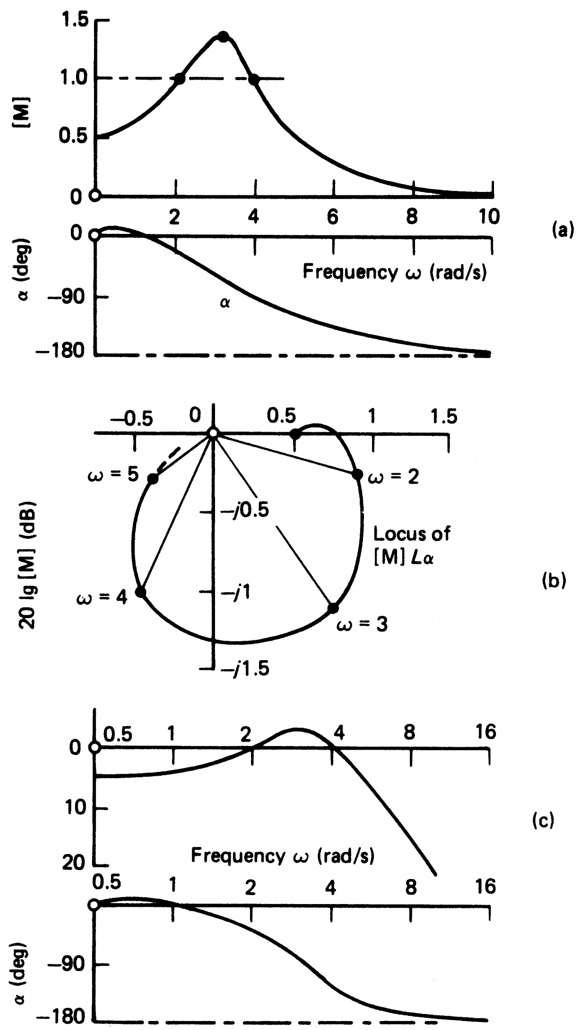


Figure 3.34 Harmonic response

short circuits the output terminals so that $V_o/V_i = 0$. At intermediate frequencies the gain $|M|$ rises to a peak at $\omega = 3.3$ rad/s and thereafter falls toward zero. The phase angle α is small and positive below $\omega = 1$, being always negative thereafter, to become $-180°$ at infinite frequency.

Nyquist diagram The Nyquist diagram is a polar plot of $|M| \angle\alpha$ over the frequency range (*Figure 3.34*(*b*)), for an input $V_i = 1 + j0$. The plot is particularly useful for feedback systems. If the open-loop transfer function is plotted, and in the direction of increasing ω it encloses the point $(-1 + j0)$, then when the loop is closed the system will be unstable as the output is more than enough to supply a feedback input even when $V_i = 0$. The Nyquist criterion for stability is therefore that the point $(-1 + j0)$ shall not be enclosed by the plot.

Bode diagram The Bode diagram for the system shown in *Figure 3.33* is *Figure 3.34*(*a*) redrawn with logarithmic ordinates of $|M|$ and a logarithmic scale of ω. Normally the ordinates are expressed as a *gain* $20 \log |M|$ in decibels. For the example being considered, $M = 0.5$ for very low frequencies, so that $20 \log |M| = -6$ dB; for $\omega = 3.3$ the amplitude of M is 1.4 and the corresponding gain is +2.9 dB; and at the two frequencies when the output and input magnitudes are the same, $M = 1$ and $20 \log (1) = 0$ dB. All these are shown in the Bode diagram (*Figure 3.34*(*c*)). On the logarithmic frequency scale, equal ratios of ω are separated by equal distances along the horizontal axis. If successive values 0.5, 1, 2, 4, ..., are marked in equidistantly, their successive ratios 1/0.5, 2/1, ..., are all equal to 2, so that each interval is a *frequency octave*. Correspondingly the equispaced frequencies 0.1, 1, 10, ..., express a *frequency decade*.

The phase-angle plot is drawn in degrees to the same logarithmic scale of frequency.

An advantage of the Bode plot is the ease with which system functions can be built up term by term. The product of complex operators is reduced to the addition of the logarithms of their moduli and phase angles; similarly the quotient is reduced to subtraction. If the system function can adequately be expressed in *simple* terms, the Bode diagram can be rapidly assembled. Such terms are listed below.

(1) $j\omega$: represented by a line through $\omega = 1$ and rising with frequency at 6 dB per octave or 20 dB per decade, and with a constant phase angle $\alpha = 90°$.
(2) $1/j\omega$: as for $j\omega$, but falling with frequency, and with $\alpha = -90°$.
(3) $1 + j\omega T$: a straight line of zero gain for frequencies up to that for which $\omega T = 1$, and thereafter a second straight line rising at 6 dB per octave; the change of direction occurs at the *break point* (*Figure 3.35*(*a*)).
(4) $1/(1 + j\omega T)$: as for $1 + j\omega T$, except that after the break point the gain drops with frequency at 6 dB per octave.

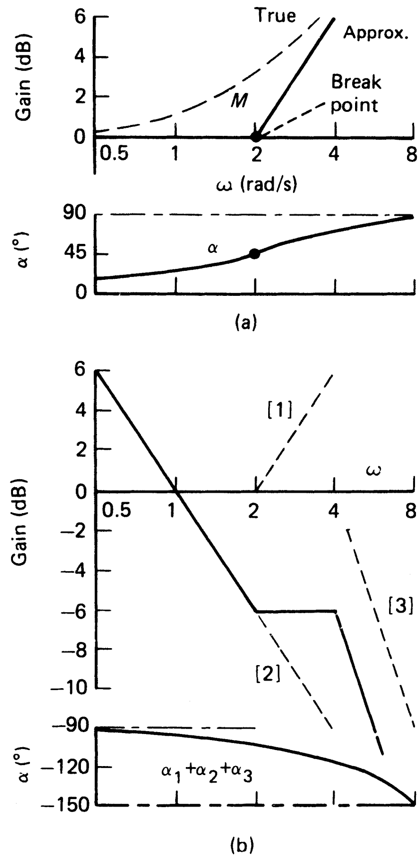


Figure 3.35 Bode diagrams

In *Figure 3.35*(*a*) the true gain shown by the broken curve is approximated by the two straight lines meeting at the break point. The approximate and true gains, and the phase angles, are given in *Table 3.5* for the term $1 + j\omega T$. The error in the gain is 3 dB at the break point, and 1 dB at one-half and twice the break-point frequency, making correction very simple.

The uncorrected Bode plot for the system function

$$KG(j\omega) = K\frac{(1 + j\omega 0.5)}{j\omega(1 + j\omega 0.25)^2} = K\frac{[1]}{[2][3]}$$

is shown in *Figure 3.35*(*b*). Term [1] is the same as in *Figure 3.35*(*a*). Term [2] is a straight line running downward

Table 3.5 Gain and phase angle for $1+j\omega T$

ωT	*Gain (dB)*		*Angle (°)*	ωT	*Gain (dB)*		*Angle (°)*
	Approx.	*True*			*Approx.*	*True*	
0	0	0.0	0	2	6	7.0	63.5
0.01	0	0.00	0.5	4	12	12.3	76
0.1	0	0.04	5.7	8	18	18.1	83
0.25	0	0.26	14	10	20	20.0	84
0.5	0	1.0	26.5	16	24	24.0	86.5
1.0	0	3.0	45	100	40	40.0	89.5

through $\omega = 1$ with a slope of 6 dB per octave. Term [3] has a break point at $\omega = 4$, but as it is a squared term its slope for $\omega > 4$ is 12 dB per octave. The full-time plot of gain is obtained by direct superposition. The effect of the constant K is to lift the whole plot upward by 20 log (K). The summed phase angles approach $-90°$ at zero frequency and $-180°$ at infinite frequency.

Nichols diagram The Nichols diagram resembles the Nyquist diagram in construction, but instead of phasor values the magnitudes are the log moduli. The point $(-1 + j0)$ of the Nyquist diagram becomes the point (0 dB, $\angle -180°$). The Nichols diagram is used for determining the closed-loop response of systems.

3.2.16 Non-linearity

A truly linear system, in which *effect* is in all circumstances precisely proportional to *cause*, is a rarity in nature. Yet engineering analyses are most usually based on a linear assumption because it is mathematically much simplified, permits of superposition and can sometimes yield results near enough to reality to be useful. If, however, the non-linearity is a significant property (such as magnetic saturation) or is introduced deliberately for a required effect (as in rectification), a non-linear analysis is essential. Such analyses are mathematically cumbersome. No general method exists, so that *ad hoc* techniques have been applied to deal with specific forms of non-linearity. The treatment depends on whether a steady-state or a transient condition is to be evaluated.

3.2.16.1 Techniques

Some of the techniques used are: (i) step-by-step solution, graphical or by computation; (ii) linearising over finite intervals; (iii) fitting an explicit mathematical function to the non-linear characteristic; and (iv) expressing the non-linear characteristic as a power series.

Step-by-step solution Consider, as an example, the growth of the flux in a ferromagnetic-cored inductor in which the inductance L is a function of the current i in its N turns. Given the flux magnetomotive-force (m.m.f.) characteristic, and the (constant) resistance r, the conditions for the sudden application of a constant voltage V are given by

$$V = ri + \mathrm{d}(Li)/\mathrm{d}t \simeq ri + N(\Delta\Phi/\Delta\tau)$$

which is solved in suitable steps of Δt, successive currents i being evaluated for use with the magnetic characteristic to start the next time-interval.

Linearising A non-linear characteristic may be approximated by a succession of straight lines, so that a piecemeal set of linear equations can be applied, 'matching' the conditions at each discontinuity.

For 'small-signal' perturbations about a fixed quiescent condition, the mean slope of the non-linear characteristic around the point is taken and the corresponding parameters derived therefrom. Oscillation about the quiescent point can then be handled as for a linear system.

Explicit function For the resistance material in a surge diverter the voltage–current relationship $v = ki^x$ has been employed, with x taking a value typically between 0.2 and 0.3.

The resistance–temperature relationship of a thermistor, in terms of the resistance values R_1 and R_2 at corresponding absolute temperatures T_1 and T_2 takes the form

$$R_2 = R_1 \exp(k/T_2 - k/T_1)$$

Several functions, such as $y = a \sinh(bx)$, have been used as approximations to magnetic saturation excluding hysteresis. A static-friction effect, of interest at zero speed in a control system, has been expressed as $y = k(\operatorname{sgn} x)$, i.e. a constant that acts against the driving torque.

Series A typical form is $y = a_0 + a_1 x + a_2x^2 + \cdots + a_nx^n$, where the coefficients a are independent of x. Such a series may have a restricted range, and the powers limited to even orders if the required characteristic has the same shape for both negative and positive y. A second-degree series $y = a_0 + a_1x + a_2x^2$ can be fitted through any three points on a given function of y, and a third-degree expression through any four points. However, the prototype characteristic must not have any discontinuities.

Rational-fraction expressions have also been developed. The open-circuit voltage of a small synchronous machine in terms of the field current might take the form $v = (27 + 0.006i)/(1 + 0.03i)$. Similarly, the magnetisation curve of an electrical sheet steel might have the B–H relationship

$$H = B(426 - 760B + 440B^2)/(1 - 0.80B + 0.17B^2)$$

with hysteresis neglected. An exponential series

$$B = a[1 - \exp(-bH)] + c[1 - \exp(-dH)] + \cdots + \mu_0 H$$

has been suggested to represent the magnetisation characteristic of a machine, the final term being related to the air gap line.

Non-linear characteristics *Figure 3.36* shows some of the typical relations $y = f(x)$ that may occur in non-linear systems. Not all are analytic, and some may require step-by-step methods.

The *simple* relations shown are: (a) response depending on direction, as in rectification; (b) skew symmetry, showing the effect of saturation; and (c) negative-slope region, but with y univalued.

The *complex* relations are: (d) negative-slope region, with y multivalued; (e) build-up of system with hysteresis, unsaturated; (f) toggle characteristic, typical of idealised saturated hysteresis; and (g) backlash, with y taking any value between the characteristic limit-lines.

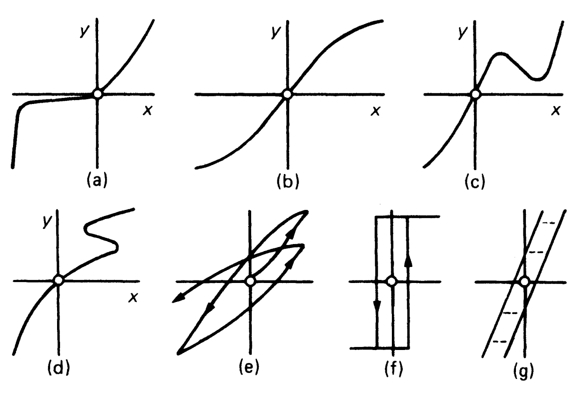


Figure 3.36 Typical non-linear characteristics

3.2.16.2 Examples

A few examples of non-linear parameters and techniques are given here to illustrate their very wide range of interest.

Resistors Thermally sensitive resistors (thermistors) may have positive or negative resistance-temperature coefficients. The latter have a relation between resistance R and absolute temperature T given by $R_2 = R_1 \exp[b(1/T_2 - 1/T_1)]$. They are made from oxides of the iron group of metals with the addition of small amounts of ions of different valency, and are applied to temperature measurement and control. Thermistors with a positive resistance-temperature coefficient made from monocrystalline barium titanate have a resistance that, for example, increases 100-fold over the range 50–100°C; they are used in the protection of machine windings against excessive temperature rise.

Voltage-sensitive resistors, made in disc form from silicon carbide, have a voltage–current relationship approximating to $v = ki^{\beta}$, where β ranges from 0.15 to 0.25. For $\beta = 0.2$ the power dissipated is proportional to v^6, the current doubling for a 12% rise in voltage.

Inductors The current in a load fed from a constant sinusoidal voltage supply can be varied over a wide range economically by use of a series inductor carrying an additional d.c.-excited winding to vary the saturation level and hence the effective inductance. The core material should have a flux-m.m.f. relationship like that in *Figure 3.36(f)*. Grain-oriented nickel and silicon irons are suitable for the inductor core. A related phenomenon accounts for the in-rush current in transformers.

Describing function In a non-linear system a sinusoidal drive does not produce a sinusoidal response. The describing-function technique is devised to obtain the *fundamental-frequency* effect of non-linearity under steady-state (but not transient) conditions.

Consider a stimulus $x = h \cos \omega t$ to give a response $y = f(t)$. As non-linearity inevitably introduces harmonic distortion, y can be expanded as a Fourier series (Section 1.2.5) to give

$$y = a_0 + a_1 \cos \omega t + a_2 \cos 2\omega t + \cdots$$
$$+ b_1 \sin \omega t + b_2 \sin 2\omega t + \cdots$$

The components $a_1 \cos \omega t$ and $b_1 \sin \omega t$ are regarded as the 'true' response, with a gain factor $(a_1 + jb_1)/h$, the other terms being the distortion. The gain factor is the describing function. Let $y = ax + bx^2$ with $x = h \cos \omega t$; applying the expansion gives the fundamental-frequency term $y = (a + \frac{3}{4}bh^2)h \cos \omega t$. The describing function is therefore $a + \frac{3}{4}bh^2$, which is clearly dependent on the magnitude h of the input. Thus the technique consists in evaluating the Fourier series for the output waveform for a sinusoidal input, and finding therefrom the magnitude and phase angle of the fundamental-frequency response.

Ferroresonance The individual r.m.s. current–voltage characteristics of a pure capacitor C and a ferromagnetic-cored (but loss-free) inductor L for a constant-frequency sinusoidal r.m.s. voltage V are shown in *Figure 3.37*. With L and C in series and carrying a common r.m.s. current I, the applied voltage is $V = V_L - V_C$. At low voltage V_L predominates, and the $I - V$ relationship is the line 0P, with I lagging V by 90°. At P, with $V = V_0$ and $I = I_0$, the system is at a limit of stability, for an increase in V results in a reduction in $V_L - V_C$. At a current level Q the difference is zero. The current therefore 'jumps' from I_0 to a higher level I_1 (point R),

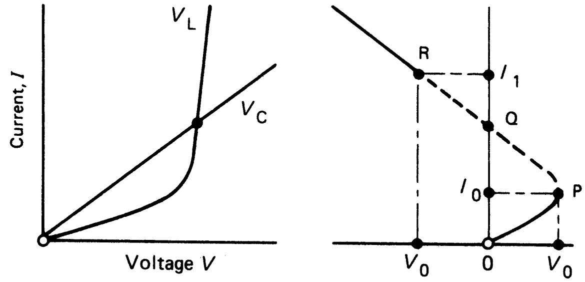


Figure 3.37 Ferroresonance

still with $V = V_0$. During the rapid rise there is an interchange of stored energy, and for $V > V_0$ the circuit is capacitive. When V is reduced from above to below V_0, a sudden current jump from I_1 to I_0 occurs. A comparable jump phenomenon takes place for a parallel connection of C and L.

Phase-plane technique 'Phase' here means 'state' (as in the solid, liquid and vapour 'phases' of water). The phase-plane technique can be used to elucidate non-linear system behaviour graphically. *Figure 3.38* shows a circuit of series R, L and C with a drive having the voltage–current relationship $v = -ri + ai^3$. Then, with constant circuit parameters,

$$L(di/dt) + (R - r + ai^2)i + q/C = 0$$

where q is the time-integral of i. The presence of L and C indicates the possibility of oscillation. The middle ('damping') term can be negative for small currents (increasing the oscillation amplitude) but positive for larger currents (reducing the amplitude). Hence the system seeks a constant amplitude irrespective of the starting condition. The $q - i$ phase-plane loci show the stable condition as related to the degree of drive non-linearity (indicated by the broken curves). With suitable scales the locus for minor non-linearity is circular, indicating near-sinusoidal oscillation; for major drive non-linearity, however, the locus shows abrupt changes and an approach to a 'relaxation' type of waveform.

Isoclines A non-linear second-order system described by

$$d^2y/dt^2 + f(y, dy/dt) + g(y, dy/dt)y = 0$$

can be represented at any point in the phase plane having the co-ordinates y and dy/dt, representing, for example, position

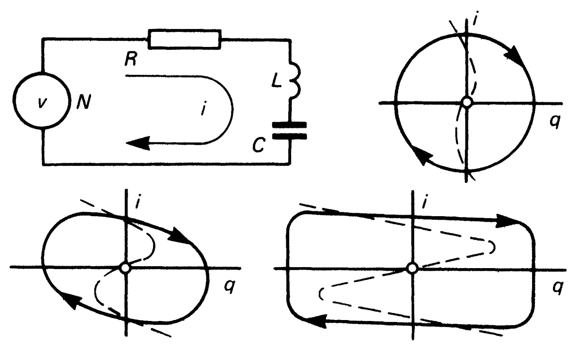


Figure 3.38 Phase-plane trajectories for an oscillatory circuit

and velocity, or charge and current. By writing $\mathrm{d}y/\mathrm{d}t = z$ and eliminating time by division, we obtain a first-order equation relating z and y:

$$\mathrm{d}z/\mathrm{d}y = -[\mathrm{f}(y,z)z + \mathrm{g}(y,z)y]/z$$

Integration gives phase-plane trajectories that everywhere satisfy this equation, starting from any initial condition. If $\mathrm{d}z/\mathrm{d}y$ cannot be directly integrated, it is possible to draw the trajectories with the aid of *isoclines*, i.e. lines along which the *slope* of the trajectory is constant. Make $\mathrm{d}z/\mathrm{d}y = m$, a constant; then $-mz = \mathrm{f}(y, z)z + \mathrm{g}(y, z)y$. Since for $z = 0$ the slope m is infinite (i.e. at right angles to the y axis) the trajectories intersect the horizontal y axis normally, except at singular points.

Consider a linear system with an undamped natural frequency $\omega_n = 1$ and a damping coefficient $c = 0.5$. For zero drive

$$\mathrm{d}^2y/\mathrm{d}t^2 + \mathrm{d}y/\mathrm{d}t + y = 0 \quad \text{or} \quad \mathrm{d}z/\mathrm{d}t = -(z+y)\Leftarrow$$

if z is written for $\mathrm{d}y/\mathrm{d}t$. Dividing the second equation by z and equating it to a constant m gives

$$m = \mathrm{d}z/\mathrm{d}y = -(z+y)/z \text{ or } z/y = -1/(1+m)\Leftarrow$$

representing a family of straight lines with the associated values

m	-4	-2	-1	0	1	2	4	$\infty\Leftarrow$
z/y	$\frac{1}{3}$	1	∞	-1	$-\frac{1}{2}$	$-\frac{1}{3}$	$-\frac{1}{5}$	0

Draw the z/y axes on the phase plane (*Figure 3.39*) marked with short lines of the appropriate slope m. Then, starting at any arbitrary point, a trajectory is drawn to cross each axis at the indicated slope. With no drive, all trajectories approach, and finally reach, the origin after oscillations in a counter-clockwise direction; for a steady drive V, the only difference is to shift the vortex to V on the y-axis. The approach to O or V represents the decaying oscillation of the system and its final steady state. Because $\mathrm{d}t = \mathrm{d}y/z$, the finite difference $\Delta t = \Delta y/z$ gives the time interval between successive points on a trajectory.

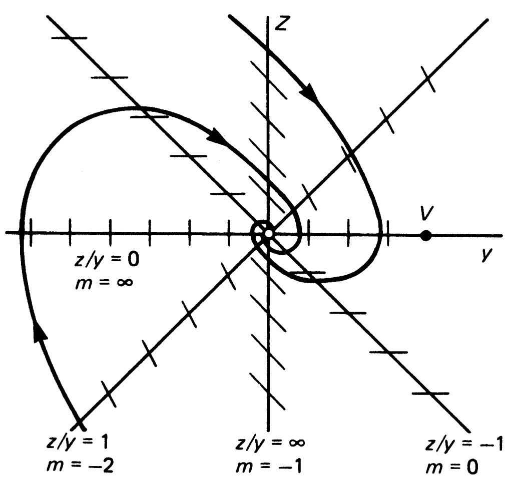


Figure 3.39 Phase-plane trajectories

3.3 Power-system network analysis

3.3.1 Conventions

Modern power-system analyses are based mainly on nodal equations scaled to a per-unit basis, with a particular convention for the sign of reactive power.

3.3.1.1 Per-unit basis

The total apparent power in a three-phase circuit ABC with phase voltages V_a, V_b, V_c, currents I_a, I_b, I_c and phase angles between the associated voltage and current phasors of θ_a, θ_b, θ_c is

$$S = (P_a + P_b + P_c) + \mathrm{j}(Q_a + Q_b + Q_c) = P + \mathrm{j}Q$$

where for phase A the active power is $P_a = V_a I_a \cos\theta_a$ and the reactive power is $Q_a = V_a I_a \sin\theta_a$. Corresponding expressions apply for phases B and C.

If the phases are balanced, all three have the same scalar voltage V and current I, and all phase angles are θ. Then $S = 3(VI\cos\theta + \mathrm{j}VI\sin\theta)$. This can be written in the form

$$\frac{S}{k} = \left[\left(\frac{3}{a}\frac{V}{b}\frac{I}{c}\frac{\cos\theta}{d}\right) + \mathrm{j}\left(\frac{3}{a}\frac{V}{b}\frac{I}{c}\frac{\sin\theta}{d}\right)\right]$$

where k, a, b, c and d are scaling factors. It is customary to choose $a = 3$ and $d = 1$, leaving c and b, one of which is assigned an independent value while the other takes a value depending on the overall scaling relation $k = abcd$.

In normal operating conditions the scalar voltage V approximates to the rated phase voltage V_R; hence b is taken as V_R so that V/b is the voltage in per unit of V_R. Defining the scaled variables as S_{pu}, V_{pu} and I_{pu}, the *total* apparent power is

$$S_{pu} = (V_{pu}I_{pu}\cos\theta) + \mathrm{j}(V_{pu}I_{pu}\sin\theta)\Leftarrow$$

which is an equation in one-phase form, justifying the use of single-line schematic diagrams to represent three-phase power circuits.

The scaling factors are termed *base* values; i.e. k is the base apparent power, b is the base phase voltage and c is the base current. These definitions imply further base values for impedance and admittance, namely $Z_{base} = b/c$ and $Y_{base} = c/b$.

If the line-to-line voltage V_l is used, then in the foregoing scaling equation $3/a$ and V/b becomes $\sqrt{3}/a'$ and V_l/b'. Choosing $a' = \sqrt{3}$ and b' as rated line voltage leaves S/k unchanged. Note, however that: (i) θ remains as the angle between *phase* voltage and current; and (ii) in the per-unit equation $S = VI^*$ the voltage V is the per-unit *phase* voltage, not the line-to-line voltage (although numerically both have the same per-unit value).

3.3.1.2 Reactive power convention

Reactive power may be lagging or leading. The common convention is to consider lagging reactive power flow to be positive, as calculated from the product of the voltage and current-conjugate phasors: thus $S = VI^* = P + \mathrm{j}Q$, with Q a positive number for a lagging power factor condition. For a leading power factor, Q has the same flow direction but is numerically negative. As a consequence, an inductor *absorbs*, but a capacitor *generates*, lagging reactive power, as shown in *Figure 3.40*.

Note: system engineers refer, for brevity, to the flow of 'power' and 'vars', meaning active power and reactive power, respectively.

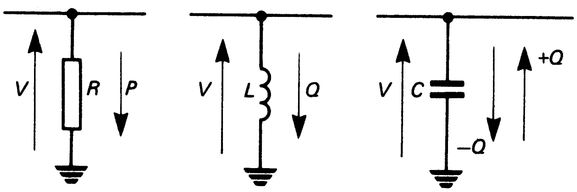


Figure 3.40 Power taken by resistive, inductive and capacitive loads

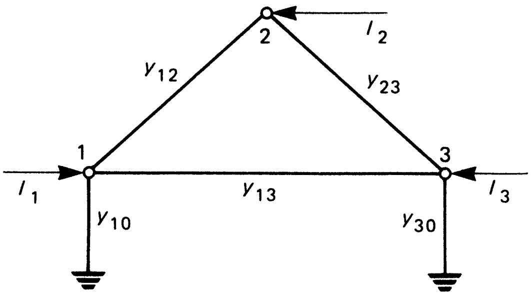


Figure 3.41 Sample network

3.3.1.3 Nodal-admittance equations

The nodal-admittance equations derive from the node-voltage method of analysis in Section 3.2.3. For the network in *Figure 3.41*, currents I_1, I_2 and I_3 are injected respectively into nodes 1, 2 and 3. The loads are linked by branches of admittance y_{12}, y_{23} and y_{31}, and to the earth or external reference node r by branches of admittance y_{10} and y_{30}. Assuming that the node voltages V_1, V_2 and V_3 are expressed with reference to r and that $V_r = 0$, then writing the Kirchhoff current equations and simplifying gives

$$(y_{10} + y_{12} + y_{13})V_1 - y_{12}V_2 - y_{13}V_3 = I_1$$
$$-y_{21}V_1 + (y_{21} + y_{23})V_2 - y_{23}V_3 = I_2$$
$$-y_{31}V_1 - y_{32}V_2 + (y_{30} + y_{31} + y_{32})V_3 = I_3$$

Cast into matrix form, these equations become

$$\begin{pmatrix} y_{10} + y_{12} + y_{13} & -y_{12} & -y_{13} \\ -y_{21} & y_{21} + y_{23} & -y_{23} \\ -y_{31} & -y_{32} & y_{30} + y_{31} + y_{32} \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \\ V_3 \end{pmatrix} = \begin{pmatrix} I_1 \\ I_2 \\ I_3 \end{pmatrix}$$

This can be abbreviated to the nodal-admittance matrix equation $\boldsymbol{YV} = \boldsymbol{I}$, matrix $\boldsymbol{Y}$ being known as the nodal-admittance matrix. The relationship between the branch elements and the corresponding matrix elements can be seen by inspection.

For some purposes the alternative impedance matrix equations are used: *viz.* $\boldsymbol{ZI} = \boldsymbol{V}$, where $\boldsymbol{Z} = \boldsymbol{Y}^{-1}$ is the nodal-impedance matrix. However, evaluating the inverse of $\boldsymbol{Y}$ is more complicated than finding $\boldsymbol{Y}$, and the admittance form may be considered as the primary (or given) form.

3.3.2 Load-flow analysis

Load-flow analysis is the solution of the nodal equations, subject to various constraints, to establish the node voltages. At the same time generator power outputs, transformer tap settings, branch power flows and powers taken by voltage-sensitive loads (including reactive power compensators) are determined.

3.3.2.1 Problem description

As indicated in *Figure 3.41*, the elements of a power system can be represented either as equivalent branches with appropriate admittance incorporated into the matrix $\boldsymbol{Y}$, or as equivalent current sources added to the matrix $\boldsymbol{I}$.

Transmission lines and cables These are represented by their series admittance and shunt (charging) susceptance. These parameters are actually distributed quantities, but are taken into account by Π (or, occasionally, T) equivalent networks (Section 3.2.5).

Transformers These are modelled by equivalent circuits with an ideal transformer in series with a leakage admittance (*Figure 3.42(a)*). With two terminals connected to a common reference point (earth) the circuit reduces to that in *Figure 3.42(b)*: the ideal transformer with a turns ratio $(1 + t)/1$ is replaced by an equivalent Π. The tap setting t represents the per-unit of nominal turns ratio: e.g. $t = \pm 0.05$ for ±5% taps.

Loads These can be represented either as equivalent admittances Y_{k0} connected between bus-bars and earth, or as current sources. If the load demand at bus-bar k is $S_k = P_k + jQ_k$, then the equivalent admittance at voltage V_k is found from

$$S_k = V_k I_k^* = |V_k|^2 Y_{k0}^* \quad \text{or} \quad Y_{k0} = S_k^*/|V_k|^2$$

where $|V_k|$ is assumed to be 1.0 p.u. (i.e. rated voltage). The equivalent admittance is incorporated into the corresponding diagonal element of the nodal-admittance matrix. In the current-source representation, the current I_k is substituted into the current column-matrix $\boldsymbol{I}$ by a fixed power requirement S_k and the (unknown) voltage V_k, where $I_k = S_k^*/V_k^*$.

Generator units or stations can likewise be represented by current sources, but usually the bus-bar to which a generator is connected has a controlled voltage. At a bus-bar m the requirement would be for specified values of active power P_m and voltage $|V_m|$, with the reactive power Q_m to be determined.

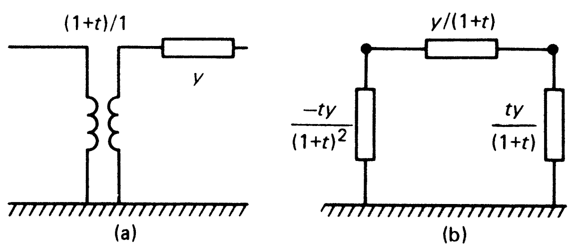


Figure 3.42 Transformer equivalent circuit

Slack bus-bar In a load-flow study, the total active power supplied cannot be specified in advance because the loss in the supply network will not be known. Further, in an n-bus-bar network, there are n complexor equations involving $2n$ real-number equations. However, there are $2n+2$ unknowns. To reduce this number to $2n$ it is the practice to specify the voltage of one bus-bar in both magnitude and phase angle. This is termed the *slack bus-bar*, to which the chosen *slack generator* is connected. The slack-bus-bar equation can now be removed from the solution process and, when all other voltages have been determined, the slack-bus-bar generation can be found. For a slack bus-bar k, for example, the generation S_k is found from

$$S_k = V_k \Sigma_m (Y_{km}^* V_m^*) \Leftarrow$$

3.3.2.2 Network solution process

Solution of the matrix equation is now sought. As it embodies several simultaneous equations, the solution has to be iterative. The two main procedures are the Gauss–Seidel and the Newton–Raphson methods. An important consideration in the computation process is the rate of convergence.

3.3.2.3 Gauss–Seidel procedure

This early (and still effective) technique resembles the over-relaxation method used in linear algebra. Consider a four bus-bar network described by the equations

$$Y_{11}V_1 + Y_{12}V_2 + Y_{13}V_3 + Y_{14}V_4 = S_1^*/V_1^*$$
$$Y_{21}V_1 + Y_{22}V_2 + Y_{23}V_3 + Y_{24}V_4 = S_2^*/V_2^*$$
$$Y_{31}V_1 + Y_{32}V_2 + Y_{33}V_3 + Y_{34}V_4 = S_3^*/V_3^*$$
$$Y_{41}V_1 + Y_{42}V_2 + Y_{43}V_3 + Y_{44}V_4 = S_4^*/V_4^*$$

where $Y_{12} = -y_{12}$, etc. Let bus-bar 1 be chosen as the slack bus-bar, and let V_1 be $1 + \mathrm{j}0$. Then the remaining three equations are to be solved for V_2, V_3 and V_4. The method adopted is one of successive estimation.

First, the equations are rearranged by extracting the diagonal terms $Y_{11}V_1, Y_{22}V_2, \ldots,$ and transferring all other terms to the right-hand side. Each equation is then divided by the diagonal admittance element (Y_{11}, $Y_{22}, \ldots$).

If $V_k^{(p)}$ denotes the pth estimate of V_k and the equations are solved in the sequence $2-3-4-2-3-\ldots$, then an iterative process is the following:

$$V_2^{(p+1)\Leftarrow} = -(Y_{21}/Y_{22})V_1 + 0 - (Y_{23}/Y_{22})V_3^{(p)\Leftarrow} - (Y_{24}/Y_{22})V_4^{(p)} + S_2^*/Y_{22}V_2^*(p)$$

$$V_3^{(p+1)\Leftarrow} = -(Y_{31}/Y_{33})V_1 - (Y_{32}/Y_{33})V_2^{(p+1)} + 0 - (Y_{34}/Y_{33})V_4^{(p)} + S_3^*/Y_{33}V_3^*(p) \Leftarrow$$

$$V_4^{(p+1)\Leftarrow} = -(Y_{41}/Y_{44})V_1 - (Y_{42}/Y_{44})V_2^{(p+1)\Leftarrow} - (Y_{43}/Y_{44})V_3^{(p+1)} + 0 + S_4^*/Y_{44}V_4^*(p) \Leftarrow$$

Note that as each new estimate becomes available, it is used in the succeeding equations. Being iterative, the process of convergence can usually be assisted by the use of 'acceleration'. If $\Delta V_k^{(p)} = V_k^{(p+1)} - V_k^{(p)}$, then a new estimate can be obtained from $V_k^{(p+1)} = V_k^{(p)} + \omega \Delta V_k^{(p)}$; here ω is an accelerating factor, optimally a complex number but usually taken as real, with typical values in the range 1.0–1.6.

To terminate the successive-estimation process, various convergence tests are applied. The simplest is to examine the difference between successive voltage estimates and to stop when the maximum of $|V_k^{(p+1)} - V_k^{(p)}|$ for $k = 1, 2, \ldots, n$ is less than ϵ, a suitable small number such as 0.000 01 p.u. However, the preferred test is

$$V_k^{(p)\Leftarrow} \Sigma_m Y_{km}^* V_m^*(p) - S_k < \gamma\,\psi$$

where γ is a measure of the maximum allowable apparent-power mismatch at any bus-bar, with a value typically 0.01 p.u.

During iteration, other calculations (e.g. voltage magnitude corrections at generator bus-bars and changes in transformer tap settings) can be included. If bus-bar 2 in the example above is a generator bus-bar, then the reactive power Q_2 can be assigned an initial value, say $Q_2^{(0)} = 0$, and $V_2^{(1)}$ obtained therefrom. The voltage estimate can then be scaled to agree with the specified magnitude $|V_2|$, and $Q_2^{(1)}$ immediately calculated, prior to proceeding to the next equation.

The Gauss–Seidel procedure is well suited to implementation on a microcomputer, in which core space is limited. However, matrix-inversion techniques (as required in the Newton–Raphson procedure following) for large networks demand too much core space.

3.3.2.4 Newton–Raphson procedure

The Newton–Raphson procedure is at present the most generally adopted method. It has strong convergence characteristics and suits a wide range of problems. The method employs the preliminary terms in the Taylor series expansion (Section 1.2.4) of a set of functions of variables $\boldsymbol{V}$. The kth function is defined as

$$\mathbf{f}(V) = \mathbf{f}[V^{(p)} + \mathbf{J}(V^{(p)})(V - V^{(p)}]$$

The true set of values $\boldsymbol{V}$ is taken as given by $\boldsymbol{V} = \boldsymbol{V}^{(p)} + \gamma^{(p)}$, i.e. by the sum of an approximate set $\boldsymbol{V}^{(p)}$ and a set of error terms $\gamma^{(p)}$. Then, taking the first two terms of the Taylor expansion,

$$\mathbf{f}(V) = \mathbf{f}[V^{(p)} + \mathbf{J}(V^{(p)})(V - V^{(p)})] \Leftarrow$$

whence

$$\mathbf{J}(V^{(p)})\gamma^{(p)} = -\mathbf{f}(V^{(p)}) \Leftarrow$$

Matrix $\mathbf{J}(V^{(p)})$ is the Jacobian matrix of first derivatives of the functions $\mathbf{f}(V)$. Voltage estimates $V^{(p)}$ are used to evaluate specific matrix elements of $\mathbf{J}$; and $\gamma^{(p)}$ is the column matrix of voltage differences $\Delta V^{(p)}$ to be evaluated, these being the difference between the true and approximate values of the voltages V. Likewise, the term $-\mathbf{f}(V^{(p)})$ is the set of per-unit apparent-power differences $\Delta S^{(p)}$ between specified and calculated values, where

$$\Delta S_k{}^{(p)} = S_k - V_k{}^{(p)\Leftarrow} \Sigma_m Y_{km}^* V_m^*(p) \Leftarrow$$

The load-flow equation to be solved becomes

$$\mathbf{J}(V^{(p)})\Delta V^{(p)} = \Delta S^{(p)}$$

When $\Delta V^{(p)}$ is determined, the voltages are updated to $V^{(p+1)} = V^{(p)} + \Delta V^{(p)}$.

The polar form of the equations is most usually employed, so with $V_k = |V_k| \angle\delta_k$ and $Y_{km} = |Y_{km}| \angle \Leftarrow_{km}$ the function $\mathbf{f}_i = 0$ becomes

$$(V_i Y_{i1} V_1 \cos\beta_{i1} + V_i Y_{i2} V_2 \cos\beta_{i2} + \cdots - P_i) \Leftarrow + \mathrm{j}(V_i Y_{i1} V_1 \sin\beta_{i1} + V_i Y_{i2} V_2 \sin\beta_{i2} + \cdots - Q_i) = 0 + \mathrm{j}0$$

where $\beta_{i1} = \delta_i - \delta_1 - \Leftarrow_{i1}$ and similarly for $\beta_{i2}, \ldots$. Partial differentiation to form the terms of the Jacobian matrix, and then separation of the real and imaginary parts, gives the matrix equations to be solved. For generator bus-bars the voltage magnitudes are fixed, so that only equations in reals are needed to evaluate $\angle\,\delta$.

Generator bus-bars are often termed 'P, V' and load bus-bars referred to as 'P, Q' bus-bars, reflecting the values specified.

The Jacobian equations for the Newton–Raphson method are thus of the form

$$\begin{pmatrix} \partial \mathbf{f}_1/\partial\delta_1 & \partial \mathbf{f}_1/\partial\delta_2 & \cdots & \partial \mathbf{f}_1/\partial V_1 & \partial \mathbf{f}_1/\partial V_2 & \cdots \\ \partial \mathbf{f}_2/\partial\delta_1 & \partial \mathbf{f}_2/\partial\delta_2 & \cdots & \partial \mathbf{f}_2/\partial V_1 & \partial \mathbf{f}_2/\partial V_2 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \partial \mathbf{f}_1/\partial\delta_1 & \partial \mathbf{f}_1/\partial\delta_2 & \cdots & \partial \mathbf{f}_1/\partial V_1 & \partial \mathbf{f}_1/\partial V_2 & \cdots \\ \partial \mathbf{f}_2/\partial\delta_1 & \partial \mathbf{f}_2/\partial\delta_2 & \cdots & \partial \mathbf{f}_2/\partial V_1 & \partial \mathbf{f}_2/\partial V_2 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix} \begin{pmatrix} \Delta\delta_1 \\ \Delta\delta_2 \\ \vdots \\ \Delta V_1 \\ \Delta V_2 \\ \vdots \end{pmatrix} = \begin{pmatrix} \Delta P_1 \\ \Delta P_2 \\ \vdots \\ \Delta Q_1 \\ \Delta Q_2 \\ \vdots \end{pmatrix}$$

Written in abbreviated form, this is

$$\begin{pmatrix} \mathbf{J}_{11} & \mathbf{J}_{12} \\ \mathbf{J}_{21} & \mathbf{J}_{22} \end{pmatrix} \begin{pmatrix} \Delta\delta \\ \Delta V \end{pmatrix} = \begin{pmatrix} \Delta P \\ \Delta Q \end{pmatrix}$$

To save computer-memory space, it is usual to omit $\mathbf{J}_{12}$ and $\mathbf{J}_{12}$, an approximation that leaves two *decoupled* sets of equations. This approach, called the 'fast decoupled Newton–Raphson loadflow', is in wide use.

For any set of estimates of the voltage, the elements of the Jacobian matrix are evaluated, and the set of equations solved (using space-saving sparse-matrix programming techniques) for $\Delta\delta$ and ΔV; the values of V, ΔP and ΔQ are updated, and so on. Convergence is achieved for most networks in a few iterations.

A further development is to extend the Taylor series to the second-derivative term, when the series will terminate if Cartesian co-ordinates are employed. Iteration is more lengthy, but the convergence characteristics are more powerful. This 'second-order Newton–Raphson procedure' is gaining popularity.

3.3.3 Fault-level analysis

The calculation of three-phase fault levels in large power networks again involves solution of the nodal-admittance equations $\boldsymbol{YV} = \boldsymbol{I}$ subject to constraints.

3.3.3.1 System representation

Representation of generators and loads by fixed P, $|V|$ and P, Q requirements is not valid because of the large and sudden departure of the bus-bar voltages from their nominal values.

Passive loads are usually represented by a constant admittance, implying that the load power is proportional to the square of the bus-bar voltage. Relations $P \propto |V|^{1.2}$ and $Q \propto |V|^{1.6}$ would be more likely, but the $|V|^2$ proportionality affords a measure of demand variability and is more easily represented in the admittance matrix $\boldsymbol{Y}$.

Synchronous machines such as generators and motors are represented by a voltage source in series with an appropriate admittance. For example, at bus-bar k where the node voltage is V_k, the current could be represented by $I_k = y_k''(E_k'' - V_k)$ using the subtransient e.m.f. and admittance. The term $y_k''V_k$ can be transferred to the other side of the nodal-admittance equation in such a way that y_k'' joins any load-admittance term in the diagonal element Y_{kk}.

The network equations have now been modified to the form

$$\boldsymbol{Y}''\boldsymbol{V} = \begin{pmatrix} \boldsymbol{y}''\boldsymbol{E}'' \\ \mathbf{0} \end{pmatrix}$$

where $\boldsymbol{Y}''$ is the admittance matrix $\boldsymbol{Y}$ with diagonal elements supplemented by equivalent load admittances and machine subtransient admittances, and the right-hand-side elements are either of the type $y_k''E_k''$ or zero.

3.3.3.2 Method of solution

In three-phase short-circuit conditions the voltages V will differ from the steady-state load values, but the right-hand-side elements will, with the exception of the element corresponding to the faulted bus-bar, remain constant. Let bus-bar m be short circuited; then $V_m = 0$. Solving the remaining equations for the voltages V_i (with $i = 1, 2, \ldots, n$, $i \neq m$) by the Gauss–Seidel procedure and then substituting the voltage values obtained in the mth equation yields a new right-hand-side value of one or other of the forms

$$y''_m E''_m + I_{msc} \quad \text{or} \quad 0 + I_{msc}$$

Here I_{msc} is the three-phase per-unit short-circuit current injected into bus-bar m to make $V_m = 0$. The fault level (in megavolt-amperes (MV-A)) is then

$$V_{m(\text{prefault})} \times I_{msc}{}^* \times \text{MV-A(base)}$$

A preferred alternative uses the superposition theorem (Section 3.2.2.1). The injection of I_{msc}, when acting alone, superposes a change ΔV_m $(= -V_{m(\text{prefault})})$ at bus-bar m. The equations to be solved become

$$\begin{pmatrix} \Delta V_1 \\ \vdots \\ \boldsymbol{Y}''\Delta V_m \\ \vdots \\ \Delta V_n \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ I_{msc} \\ \vdots \\ 0 \end{pmatrix} \quad \text{or} \quad \boldsymbol{Y}''\Delta\boldsymbol{V} = \boldsymbol{I}_{sc}$$

Inversion of $\boldsymbol{Y}''$ gives $\boldsymbol{V} = \boldsymbol{Z}''\boldsymbol{I}_{sc}$, in which the mth equation is known to be $-V_m = Z_{mm}''I_{msc}$. If we assume nominal prefault voltage, i.e. $V_m = 1 + \mathrm{j}0$ p.u., then the value of the three-phase short-circuit current at bus-bar m is $I_{msc} = -1/Z_{mm}''$. The voltage at any other bus-bar k can then be found from

$$V_k = V_{k(\text{prefault})} + \Delta V_k = V_{k(\text{prefault})} + Z_{km}''I_{msc}$$

By shifting the short circuit from bus-bar to bus-bar, the fault level for each can be found from the inverses of the appropriate diagonal elements of matrix $\boldsymbol{Z}''$.

3.3.4 System-fault analysis

The analysis of *unbalanced* faults in three-phase power networks is an important application of the *symmetrical-component* method (Section 3.2.12). The procedure for given fault conditions is as follows.

(1) Obtain the sequence impedance values for all items of the plant, equipment and transmission links concerned.
(2) Reduce all ohmic impedances to a common line-to-neutral base and a common voltage.
(3) Draw a single-line connection diagram for each of the sequence components, simplifying where possible (e.g. by star–delta conversion, see Section 3.2.4.5).
(4) Calculate the z.p.s., p.p.s. and n.p.s. currents, tracing them through the network to obtain their distribution with reference to the particular values sought.

Impedance in the neutral connection to earth, and in the earth path itself, must be multiplied by 3 for z.p.s. currents when the z.p.s. connection diagram is being set up in (3) because the three z.p.s. component currents are co-phasal and flow together in the z.p.s. path.

In general, a network offers differing impedances Z_+, Z_- and Z_0 to the sequence components. In static plant (e.g. transformers and transmission lines) Z_- may be the same as Z_+, but Z_0 is always significantly different from either of the other impedances. The presence of z.p.s. currents implies that a neutral connection is involved.

3.3.4.1 Sequence networks

As an example, *Figure 3.43* shows transmission lines 4 and 5–6 linking a generating station with generators 1 (isolated neutral) and 2 (solid-earthed neutral) to a second station with generator 3 (neutral earthed through resistor R_n). The numerals are used to indicate position: e.g. Z_{1+} is the p.p.s. impedance per phase of generator 1, and Z_{60} is the z.p.s. impedance of line 6 between generator 3 and a fault at F.

The *p.p.s. network* is identical with the physical set-up of the original network (which operates with p.p.s. conditions when normally balanced and unfaulted). Each generator is a source of p.p.s. voltages only. It is here assumed that all the generators develop the same e.m.f. E_a.

The *n.p.s. network* is similar in configuration (but not usually in impedance values) to the p.p.s. system. There are, however, no source e.m.f.s: the n.p.s. voltages are 'fictitious' ones developed by the fault.

The *z.p.s. network* is radically different from the other two, being concerned with neutral connections and earth faults. The effective line impedance is that of three conductors sharing equally the total n.p.s. current. To this must be added the earth-connection and earth-path impedances multiplied by 3, to give the z.p.s. impedance Z_0.

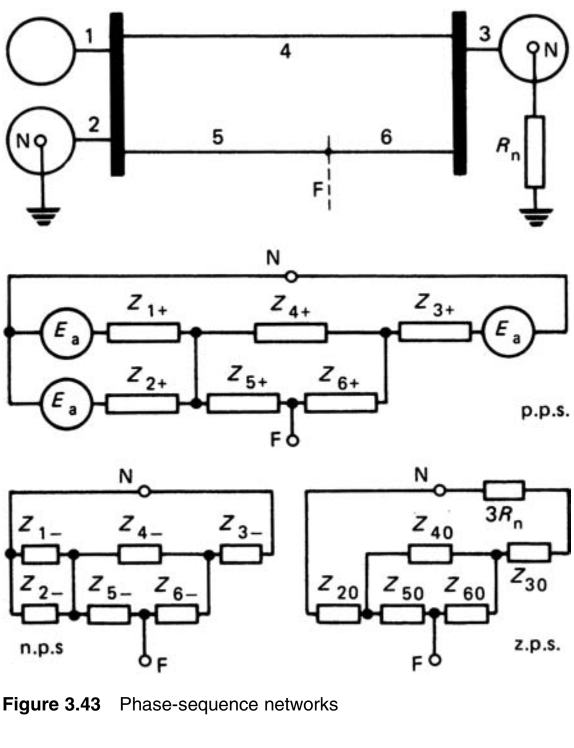


Figure 3.43 Phase-sequence networks

Typical values of Z_- and Z_0 in terms of Z_+ are

Ratio	*Generator*	*Transformer*	*Transmission link*	
			Overhead	*Cable*
Z_-/Z_+	0.6–0.7	1.0	1.0	1.0
Z_0/Z_+	0.1–0.8	1.0 or ∞	3–5	1–3

The value of Z_0 for a synchronous generator depends on the arrangement of the stator winding.

To evaluate the system when faulted, it is necessary to determine the fault currents and the voltage of the sound line(s) to earth. If the voltages and currents at the fault are V_a, V_b, V_c and I_a, I_b, I_c, respectively, the following expressions always apply:

$$V_a = E_a - I_+Z_+ - I_-Z_- - I_0Z_0; \quad I_a = I_+ - I_- + I_0$$
$$V_b = \alpha^2 E_a - \alpha^2 I_+Z_+ - \alpha I_-Z_- - I_0Z_0; \quad I_b = \alpha^2 I_+ + \alpha I_- + I_0$$
$$V_c = \alpha E_a - \alpha I_+Z_+ - \alpha^2 I_-Z_- - I_0Z_0; \quad I_c = \alpha I_+ + \alpha^2 I_- + I_0$$

where α is the 120° rotation operator (see Section 3.2.12). From the boundary conditions at the fault concerned it is possible to write three equations and to solve them for the symmetrical components I_+, I_- and I_0.

Sequence networks for some of the many transformer connections are shown in *Figure 3.44*. Further sequence networks are given in reference 1.

3.3.4.2 Boundary conditions

Three simple cases are shown in *Figure 3.45*. It is assumed that only fault currents are concerned, and that in-feed to the fault is from one direction.

(*a*) *Earth fault of resistance R_f on line A*—at the fault, $V_a = I_aR_f$ and $I_b = I_c = 0$. This leads to $I_{a0} = I_{a+} = I_{a-}$, so that the three sequence currents in phase A are identical. It follows that I_b and I_c are zero, as required. From the basic equations

$$I_{a+} = I_{a-} = I_{a0} = \tfrac{1}{3}I_a = E_a/(Z_+ + Z_- + Z_0 + 3R_f) = E_a/Z$$

and the fault current is $I_a = 3E_a/Z$. The three sequence networks are, in effect, connected in series. The component currents, and the voltages V_b and V_c, are obtained from those in phase A by application of the basic relations in Section 3.3.4.1. Each sequence current divides in the branches of its network in accordance with the configuration and impedance values.

(*b*) *Short circuit between lines B and C*—the boundary conditions are $I_a = 0$, $I_b = -I_c$ and $V_b = V_c$. As there is no connection to earth at the fault, the z.p.s. network is omitted. The p.p.s. and fault currents are

$$I_{a+} = E_a/(Z_+ + Z_-) = E_a/Z$$
$$I_b = -I_c = -j\sqrt{3}E_a/Z$$

where $Z = Z_+ + Z_-$ is the impedance of the p.p.s. and n.p.s. networks in series. The voltages to neutral at the fault are

$$V_a = E_a - I_+Z_+ - I_-Z_- = 2E_a(Z_-/Z)$$
$$V_b = V_c = E_a(Z_-/Z)$$

(*c*) *Double line–earth fault on lines B and C*—here the boundary conditions are $I_a = 0$, and $V_b = V_c = 0$. The sequence components of the fault current are

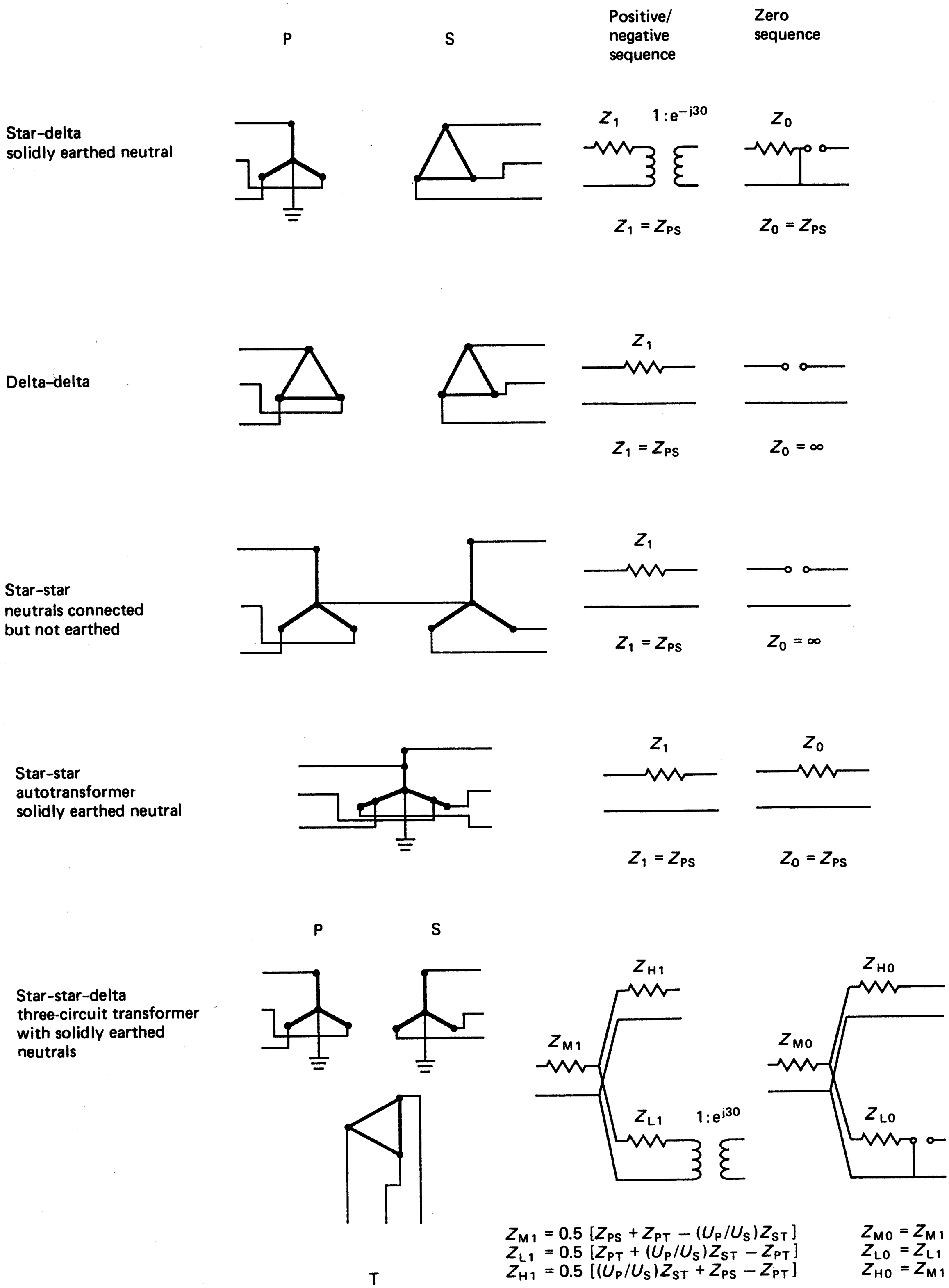


Figure 3.44 Transformer equivalent sequence networks. U_P and U_S and the 3-phase MVA ratings of winding P and S respectively

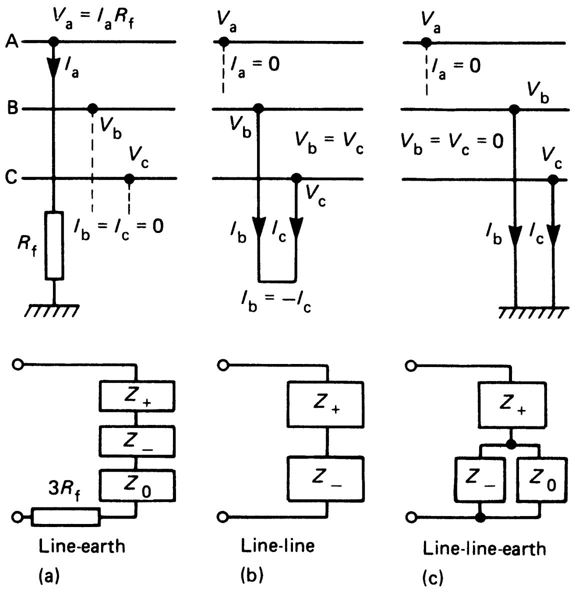


Figure 3.45 Boundary conditions at line faults

$$I_{a+} = E_a/[Z_+ + Z_-Z_0/(Z_- + Z_0)]\Leftarrow$$
$$I_{a-} = -I_{a+}[Z_0/(Z_- + Z_0)]\Leftarrow$$
$$I_{a0} = -I_{a+}[Z_-/(Z_- + Z_0)]\Leftarrow$$

The sequence networks are connected in series–parallel.

Figure 3.46 shows the interlinked phase-sequence networks where both ends feed the fault F. Conditions in (*a*), (*b*) and (*c*) correspond to those in *Figure 3.45*. Networks for a broken-conductor condition are shown in (*d*) and (*e*): the former is for a case in which both ends at the break remain insulated, while the latter applies where the conductor on side 2 falls to earth, the additional constraint involving ideal 1/1 transformers in side 2 of the combined sequence network. In more complicated cases, ideal transformers with phase-shift or with ratios other than 1/1 may be required. Evaluation of these, and of conditions involving simultaneous faults at different points and/or phases, requires matrix analysis and computer programs.

3.3.5 Phase co-ordinate analysis

The same techniques developed for the analysis of balanced networks, i.e. load-flow and three-phase fault-level analysis, can be used to develop the analysis of unbalanced faults or loads on either balanced or unbalanced networks using a phase representation of the system. The symmetrical component theory, which has formed the basis of unbalanced power system network analysis, was developed by several authors between 1912 and 1918. Although forming the basis of most modern computer-aided fault analyses, the method is limited by both balanced network assumptions and the difficulties associated with finding equivalent network models in the space described by the set of (transformed) 0, 1, 2 variables, e.g. for simultaneous faults.

The phase co-ordinate method described here uses only the primary phase *a*, *b*, *c* variables and computer-based

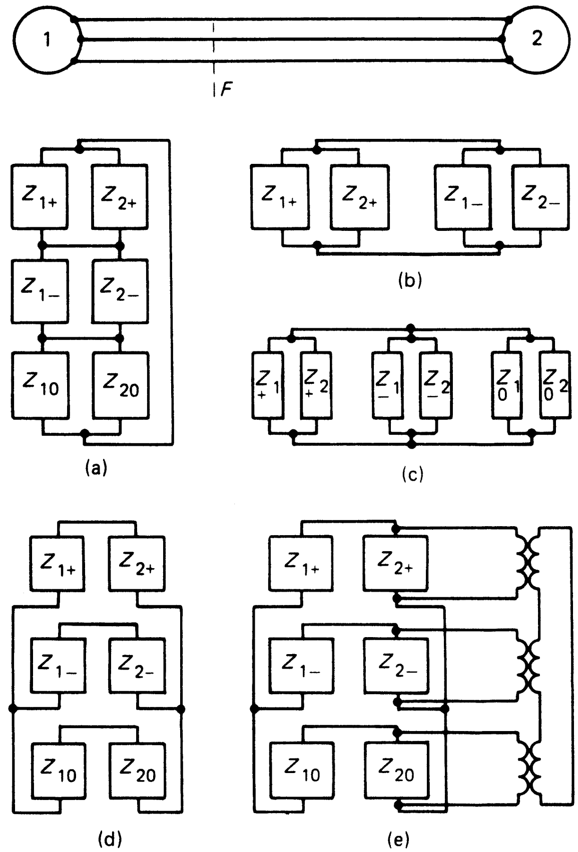


Figure 3.46 Interconnected phase-sequence networks

matrix computational methods to solve the resulting equations subject to the necessary constraints imposed by load-flow or fault analyses.[2–6]

3.3.5.1 Element representation

For brevity, when referring to the sequence frame of reference, only the zero-, positive- and negative-sequence components usually associated with Fortescue will be discussed.

General element The general network element of the power system shown in *Figure 3.47* for a three-phase system may be described in matrix form by

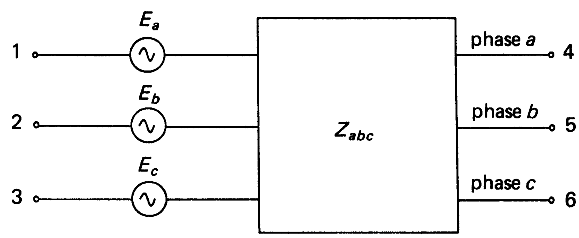


Figure 3.47 General three-phase system element

$$V_{abc} = E_{abc} + Z_{abc} I_{abc}$$

where V_{abc} represents the per-phase series voltage drops $(V_1 - V_4)$, $(V_2 - V_5)$ and $(V_3 - V_6)$, E_{abc} is the matrix of the equivalent voltage sources per phase, I_{abc} is the matrix of currents flowing per phase between nodes 1 and 4, 2 and 5, and 3 and 6, respectively, and Z_{abc} represents the passive three-phase mutually coupled network.

Defining the transformation relating the phase to the sequence components by

$$V_{abc} = T V_{012}$$

where $\alpha = 1 \angle 120°$ and, for the usual Fortescue components V_0, V_1 and V_2, will be used instead of the rather cumbersome notation V_0, V_+ and V_-; thus

$$T = \begin{pmatrix} 1 & 1 & 1 \\ 1 & \alpha^2 & \alpha \\ 1 & \alpha & \alpha^2 \end{pmatrix}$$

Noting that $T^{-1} = \frac{1}{3} T^*$, upon transformation the equivalent sequence relationship becomes

$$T V_{012} = T E_{012} + Z_{abc} T I_{012}$$

or

$$V_{012} = E_{012} + Z_{012} I_{012}$$

where

$$Z_{012} = \tfrac{1}{3} T^* Z_{abc} T \qquad (3.3)$$

and likewise

$$Y_{012} = \tfrac{1}{3} T^* Y_{abc} T$$

For the customary linear device Z_{abc} is symmetric, but without further assumptions Z_{012} is unsymmetric.

Transmission lines The series relationships along a transmission line may be represented by the partitioned equations

$$\begin{pmatrix} V_1 \\ V_2 \\ 0 \end{pmatrix} = \begin{pmatrix} Z_{11} & Z_{12} & Z_{13} \\ Z_{21} & Z_{22} & Z_{23} \\ Z_{31} & Z_{32} & Z_{33} \end{pmatrix} \begin{pmatrix} I_1 \\ I_2 \\ I_3 \end{pmatrix}$$

where V_1 and V_2 are column submatrices representing the series voltages along conductors and subconductors, respectively, carrying the currents in the submatrices I_1 and I_2. I_3 is a submatrix representing the currents in any earth wires present, which are assumed to be solidly earthed at each end. This latter assumption is normal, but unnecessary, for the solution of the system equations, because extra equations representing earth wires can be included for solution.

A matrix reduction (completion of bundling process and earth-wire removal) will then yield the equations of the equivalent phase conductors, thus giving

$$V_{abc} = Z_{abc} I_{abc}$$

The 3×3 matrix Z_{abc} represents the series impedance of the transmission line, including all the effects due to unbalanced configuration and the use of bundled conductors and earth wires. In a three-phase representation of the network this matrix may be treated in the same way as the single series impedance of the more usual one-line diagram.

By similar reasoning, the potential coefficients of the transmission line (phase and earth wires) per unit of length may be computed for any configuration. From these coefficients a shunt-reactance matrix is found, and then, by reduction, the 3×3 matrix Z_{shunt} representing the total distributed capacitive reactance over the length of the line. Inverting this matrix gives Y_{shunt} from which the Maxwell coefficients, and hence capacitances, can be determined if necessary.

Three-phase transmission-line models follow as extensions of the nominal or distributed π and T representations. In the π representation one-half of the transmission-line equivalent capacitance is connected to each end of the line.

The nodal voltages and currents injected into the busbars at each end of the transmission line are then related by

$$\begin{pmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \\ I_5 \\ I_6 \end{pmatrix} = \begin{pmatrix} Y_{abc} + \frac{1}{2} Y_{shunt} & -Y_{abc} \\ -Y_{abc} & Y_{abc} + \frac{1}{2} Y_{shunt} \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \\ V_6 \end{pmatrix}$$

The similarity between the three-phase and single-phase admittance matrices is evident—each element in the single-phase matrix is replaced by the appropriate 3×3 admittance sub-matrix.

Machine representation The general element shown in *Figure 3.47* may be taken to represent any synchronous or induction machine with a star-connected stator winding when nodes 1, 2 and 3 are short circuited. For a balanced design the voltage sources would be displaced 120° from each other and be equal in magnitude.

The phase-impedance matrix Z_{abc} can be found by simple transformation of the normally available sequence impedance matrix, as indicated in equations (3.3) and (3.4). The sequence impedance matrix here reflects the structure of the machine and the purpose of the study in the selection of the positive sequence reactance. A typical salient-pole diesel generator, for example, with segmented dampers, 14 poles and 90.7% winding pitch rated at 1340 kV-A, 3.3 kV, 50 Hz has negative and zero reactances of $X_2 = 0.274$ p.u. and $X_0 = 0.067$ p.u., respectively. In a fault-level study the machine is represented in the positive-sequence circuit by the subtransient reactance $X_d' = 0.227$ p.u. $= X_1$. The phase-impedance matrix would then be

$$Z_{abc} = \frac{1}{3} T Z_{012} T^* =$$

$$\begin{pmatrix} 0+j0.1893 & 0.0138-j0.0612 & -0.0138-j0.0612 \\ -0.0138-j0.0612 & 0+j0.1893 & 0.0138-j0.0612 \\ 0.0138-j0.0612 & -0.0138-j0.0612 & 0+j0.1893 \end{pmatrix} \qquad (3.4)$$

Note that, with $X_1 \neq X_2$, the phase-impedance matrix elements contain both positive and negative real parts, although no resistances were included in the sequence impedances.

Such a representation neglects the effective winding capacitances. These capacitances could be added, if necessary, in the same manner as the transmission-line capacitances in the three-phase representation in delta and shunt connections (a four-terminal lattice network with one terminal earthed). Neglected also is any representation of a machine neutral node. Instead of adding an appropriate row or column to Z_{abc}, the effect of any earthing reactance X_g can be included in the value of X_0, i.e. $X_0 + 3X_g$.

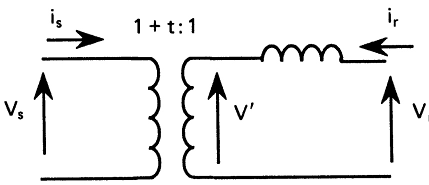


Figure 3.48 Per-unit schematic representation of a single-phase transformer

Transformer representation The third type of network element is the transformer in one-, two- and three-phase circuits with all the associated possible variations of construction and connection.

(*a*) *Derivation of equivalent-circuit model.* A single-phase representation of a transformer in per-unit form illustrated schematically in *Figure 3.48* by an ideal transformer of turns ratio $(1+t)$:1 with an equivalent leakage admittance of y per unit.

From the relationships across the ideal transformer an equivalent circuit model is as shown in *Figure 3.49* and thus represents a single-phase tapped transformer.

If nodes k and q are earthed as in the one-line diagram representation of a balanced three-phase system, the lattice circuit reduces to the equivalent π-representation, as shown in *Figure 3.50*.

The lattice equivalent of *Figure 3.49* provides an adequate model for single-phase variable-turns-ratio transformers and in combinations for three-phase star–star banks with tapped windings, but can only be used with care in banks containing delta-connected windings. In a star–delta bank of single-phase transformer units, for example, with normal turns ratio, a value of 1.0 p.u. voltage on each leg of the star winding produces under balanced conditions 1.732 p.u. voltage on each leg of the delta winding (rated line to neutral voltage as base). The structure of the bank requires in the per-unit representation an effective tapping at $\sqrt{3}$ times nominal turns ratio on the delta side, i.e. $1+t=1.732$ or $t=0.732$.

For a delta–delta or star–delta transformer with taps on the star winding, the equivalent circuit of *Figure 3.49* would have to be modified to allow for effective taps to be represented on each side. This, the general symmetrical lattice equivalent circuit of a single-phase transformer where both primary and secondary windings may have either actual or equivalent variable turns ratios α and β or both is shown in *Figure 3.51*. This single-phase transformer model can be used to assemble equivalent circuits of polyphase transformer banks, some of which are shown below

(*b*) *Star–star transformer*. For a two-circuit three-phase transformer or autotransformer connected in a star–star arrangement, the equivalent circuit is as shown in linear-graph form in *Figure 3.52*. Parallel transformer windings are taken to represent equivalent single-phase transformers. The circuit is constructed from the simple connection of three of the general circuits shown in *Figure 3.51* with taps on both windings. In practice, of course, either α or β, or both, would be 1.0 p.u.

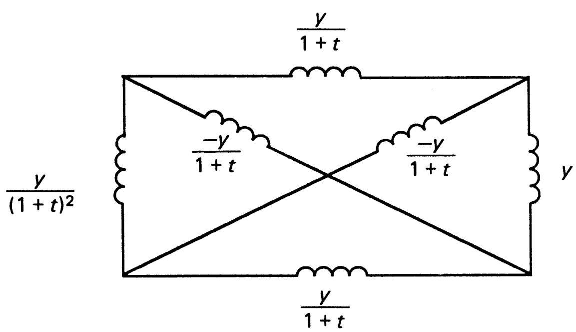


Figure 3.49 Symmetrical lattice equivalent circuit of a single-phase transformer with a variable turns ratio

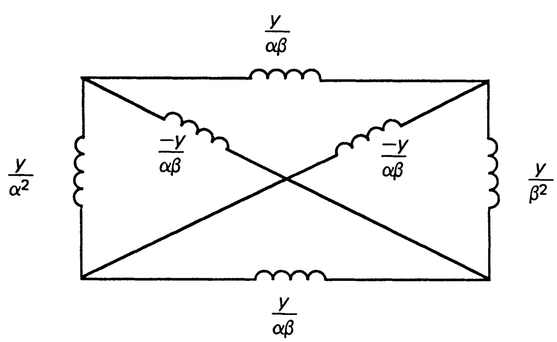


Figure 3.51 General transformer symmetrical lattice equivalent circuit with primary and secondary equivalent turns α and β per unit

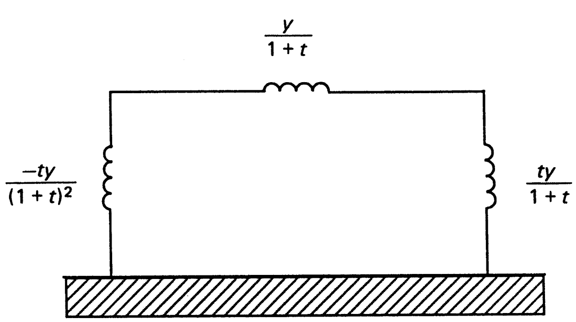


Figure 3.50 Single-phase equivalent π representation when nodes k and q are earthed

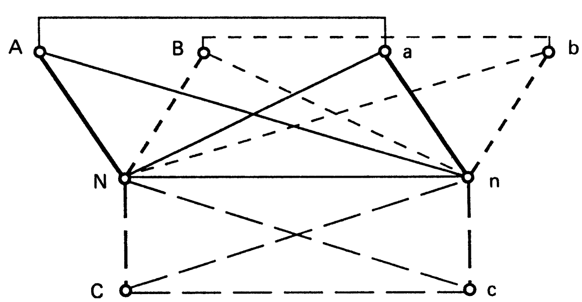


Figure 3.52 A three-phase equivalent circuit of a star–star connected transformer

Table 3.6 Connection table for the star–star transformer equivalent circuit shown in *Figure 3.52* ($\alpha = 1 + t_\alpha$ p.u.; $\beta = 1 + t_\beta$ p.u.)

Admittance	*Between nodes*
$y/\alpha^2\psi$	N–A, N–B, N–C
$y/\beta^2\psi$	n–a, n–b, n–c
$y/\alpha\beta\psi$	A–a, B–b, C–c
$-y/\alpha\beta\psi$	n–A, n–B, n–C; N–a, N–b, N–c
$3y/\alpha\beta\psi$	N–n

In a more concise form, the equivalent circuit of *Figure 3.52* may be described by the connection table given in *Table 3.6* where, for example, an admittance of value $y/\alpha^2\psi$ is connected between N and A, also N and B, etc. If the neutrals are earthed or connected together either solidly or through an impedance, the appropriate additions or deletions can be made to the circuit and corresponding terms changed in the connection table. From inspection of the circuit, the corresponding admittance matrix can be assembled with or without rows and columns for the neutral nodes, depending on the earthing arrangements.

(*c*) *Delta–delta transformer*. With the same convention (that parallel windings may be considered to represent single-phase transformers) the equivalent circuit of a delta–delta transformer may be constructed by the same principles, as shown in *Figure 3.53*. The corresponding connection table is shown in *Table 3.7*, where both windings have variable ratios. With taps on one winding only either $\alpha\psi$ or $\beta\psi$ would have the value $\sqrt{3}$, the tap value $t_{\alpha\psi}$ or $t_{\beta\psi}$ being zero accordingly.

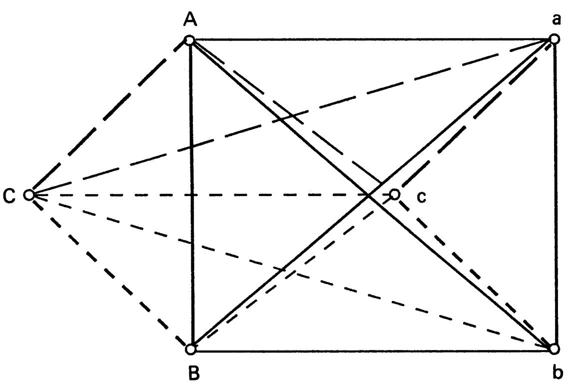


Figure 3.53 A three-phase equivalent circuit of a delta–delta connected transformer

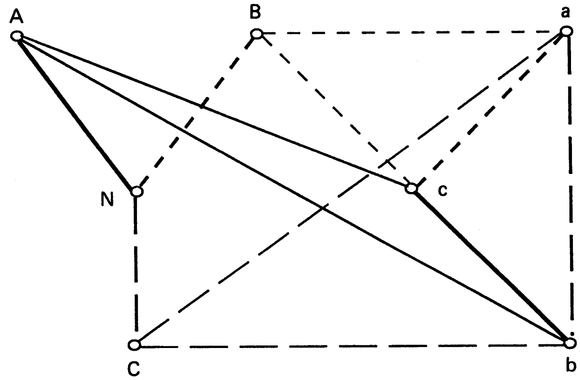


Figure 3.54 A three-phase equivalent circuit of a star–delta connected transformer

(*d*) *Star–delta transformer*. Using the same techniques, the three-phase equivalent circuit model of a star–delta transformer may be assembled and is shown in *Figure 3.54*. The convention used for numbering nodes and thus identifying opposite sides of the symmetrical lattice networks is as follows:

A–N/c–b; B–N/a–c; C–N/b–a.ψ

The connection table for *Figure 3.54* is given in *Table 3.8*. Again, for taps on one side of the transformer only, either $t_{\alpha\psi}$ or $t_{\beta\psi}$ is zero. The neutral node *N* in the table can be identified with the reference earth node if solidly earthed or extra terms can be added to the table or not, according to the earthing arrangements

(*e*) *Three-winding transformers*. With the same assumptions the analysis can be extended to three-circuit transformers and autotransformers and to any multiwinding transformer regardless of the number of circuits. Consider, for example, a star–star–delta transformer with solidly earthed neutrals. Let the star primary and secondary winding (P and S) terminals be labelled A, B, C, N and A′⇐B′, C′⇐ N′, respectively, with the delta tertiary winding (T) terminals labelled a, b, c, as shown in *Figure 3.55*.

If y_{PS}, y_{PT} and y_{ST} are the short-circuit per-unit admittances of the two windings indicated by the subscripts with the third winding open, a three-phase equivalent circuit can be assembled from paralleling one star–star and two star–delta equivalent circuits in turn. The circuit line diagram is too complex to illustrate conveniently, but with the same convention concerning the matching of parallel sides in the identification of the single-phase units, A–N with A′–N′⇐ A–N with c–b and A′–N′⇐with c–b, etc., the connection

Table 3.7 Connection table for the delta–delta transformer in the equivalent circuit shown in *Figure 3.53* ($\alpha = \sqrt{3}(1 + t_\alpha)$ p.u.; $\beta = \sqrt{3}(1 + t_\beta)$ p.u.)

Admittance	*Between nodes*
$y/\alpha^2\psi$	A–B, B–C, C–A
$y/\beta^2\psi$	a–b, b–c, c–a
$2y/\alpha\beta\psi$	A–A, B–b, C–c
$-y/\alpha\beta\psi$	A–b, B–c, C–a; a–B, b–C, c–A

Table 3.8 Connection table for the star–delta transformer in the equivalent circuit shown in *Figure 3.54* ($\alpha = 1 + t_\alpha$ p.u.; $\beta = \sqrt{3}\ (1+t_\beta)$ p.u.)

Admittance	*Between nodes*
$y/\alpha^2\psi$	A–N, B–N, C–N
$y/\beta^2\psi$	a–b, b–c, c–a
$y/\alpha\beta\psi$	A–c, B–a, C–b
$-y/\alpha\beta\psi$	A–b, B–c, C–a

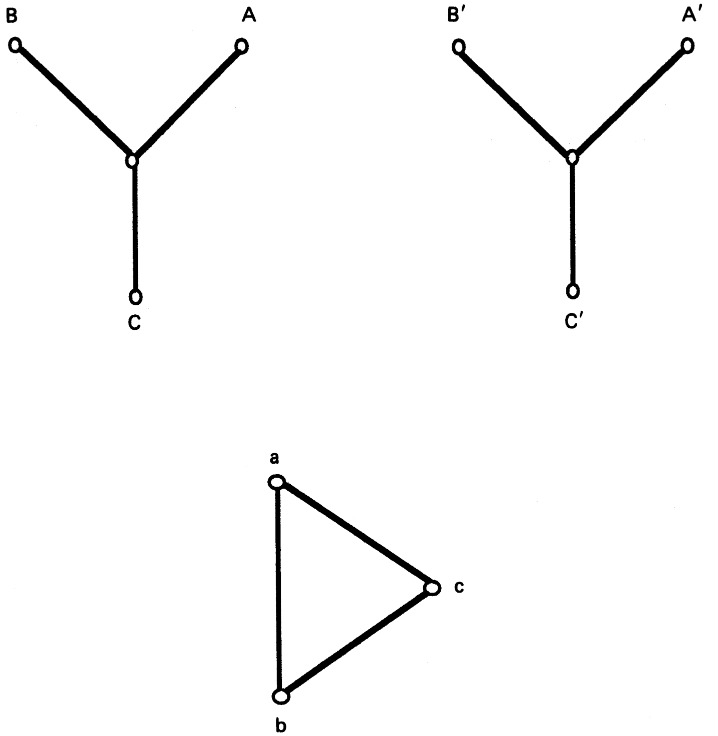


Figure 3.55 A star–star–delta transformer showing terminal markings

table will be as in *Table 3.9*, where α represents the turns ratio of winding P, β of winding S, and γ of winding T. Neutrals N and N′ are solidly earthed.

Note that, because two symmetrical lattice networks are connected to any two nodes on the same winding, a and b for example, the total admittance between these nodes is the sum of the corresponding admittances belonging to each of the two lattices between these nodes, e.g.

$$y_{PT}/\gamma^2 + y_{ST}/\gamma^2 = (y_{PT} + y_{ST})/\gamma^2$$

(*f*) *Open-delta transformer*. The validity of the equivalent-network model in representing unbalanced transformer designs is demonstrated in the analysis of the open-delta transformer. *Figure 3.56* shows a schematic circuit diagram

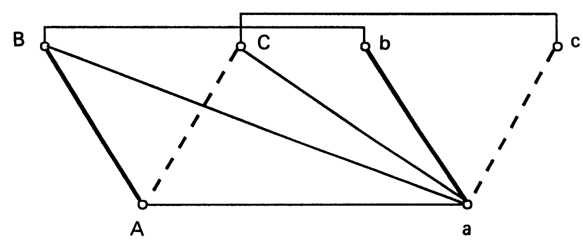


Figure 3.56 Equivalent circuit of an open-delta transformer

of the transformer with each delta open opposite nodes A and a, respectively. Connecting parallel branches of the windings by symmetrical-lattice equivalent circuits yields the connection table shown in *Table 3.10*.

Equivalent circuits for other unbalanced transformers and for transformers with different numbers of primary and secondary phases, i.e. *m*-to-*n* transformers such as Scott, Zig-zag, Vee transformers, etc., can be found in reference 5.

(*g*) *Effects of magnetising impedances*. The preceding transformer models do not account for the effects of core structure and saturation. In particular, it is noted that the phase circuits give the same representation as transformer sequence impedance circuits with a series impedance of equal value in each of the positive-, negative-, and zero-sequence circuits. More accurate representations include the transformer magnetising impedances in three-phase transformers.

Table 3.9 Connection table for a three-winding star–star–delta transformer ($\alpha = 1 + t_p$ p.u.; $\beta = 1 + t_s$ p.u.; $\gamma = \sqrt{3}(1 + t_T)$ p.u.)

Admittance	*Between nodes*
$(y_{PS} + y_{PT})/\alpha^2$	N–A, N–B, N–C
$(y_{PS} + y_{ST})/\beta^2$	N′–A′, N′–B′, N′–C′
$y_{PS}/\alpha\beta$	A–A′, B–B′, C–C′
$-y_{PS}/\alpha\beta$	N′–A, N′–B, N′–C; N–A′, N–B′, N–C′
$(y_{PT} + y_{ST})/\gamma^2$	a–b, b–c, c–a
$y_{PT}/\alpha\gamma$	A–c, B–a, C–b
$y_{ST}/\beta\gamma$	A′–c, B′–a, C′–b
$-y_{PT}/\alpha\gamma$	A–b, B–c, C–a
$-y_{ST}/\beta\gamma$	A′–b, B′–c, C′–a

Table 3.10 Connection table for the open-delta transformer with each winding open opposite nodes A and a respectively ($\alpha=\sqrt{3}(1+t_\alpha)$ p.u.; $\beta=\sqrt{3}(1+t_\beta)$ p.u.)

Admittance	*Between nodes*
$y/\alpha^2\psi$	A–B, C–A
$y/\beta^2\psi$	a–b, c–a
$y/\alpha\beta\psi$	B–b, C–c
$-y/\alpha\beta\psi$	A–b, A–c, B–a, C–a
$2y/\alpha\beta\psi$	A–a

In terms of the sequence quantities, these impedances are of particular importance in the zero-sequence networks. High values of zero-sequence voltage in shell-type transformers, and the effects of the tank walls and out-of-core return paths for zero-sequence fluxes in three-legged core-type transformers, give magnetising impedances of the same order of magnitude as the system impedances.

The transformer equivalent sequence networks can be modified to become T networks, with the magnetising impedance sequence components in the legs of the Ts. In phase coordinates the shunt impedance branch can be added likewise to the transformer single-phase model, as indicated in *Figure 3.57*. The dotted line indicates the part of the transformer short-circuit impedance placed in the lattice networks; the other part could be incorporated into a lattice network if equivalent taps are required on both sides. The equality of the series admittances has no significance.

Starting from the usual measures of these magnetising admittances in sequence terms, the three-phase equivalent circuit can be developed as follows where, for a balanced design,

$$\boldsymbol{Y}_{abc}=\tfrac{1}{3}\boldsymbol{T}\boldsymbol{Y}_{012}\boldsymbol{T}^*$$

$$=\tfrac{1}{3}\begin{pmatrix} Y_0+Y_1+Y_2 & Y_0+\alpha Y_1+\alpha^2 Y_2 & Y_0+\alpha^2 Y_1+\alpha Y_2 \\ Y_0+\alpha^2 Y_1+\alpha Y_2 & Y_0+Y_1+Y_2 & Y_0+\alpha Y_1+\alpha^2 Y_2 \\ Y_0+\alpha Y_1+\alpha^2 Y_2 & Y_0+\alpha^2 Y_1+\alpha Y_2 & Y_0+Y_1+Y_2 \end{pmatrix}$$

This phase-admittance matrix represents an equivalent delta network with connections to earth at each node where $y_1=y_2$ as shown in *Figure 3.58*.

(*h*) *Phase-shifting transformers*. Phase-shifting transformers may be represented in a similar manner from an assembly of single-phase elements, but here the single-phase elements have to be derived.

(1) *Single-phase equivalent-circuit model*—following the method described in Section 3.3.5.2, the single-phase representation may be illustrated approximately as in *Figure 3.59*, where the ideal transformer now represents

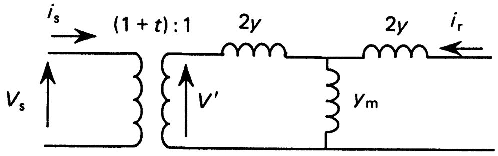


Figure 3.57 Per-unit schematic representation of a single-phase transformer with core representation

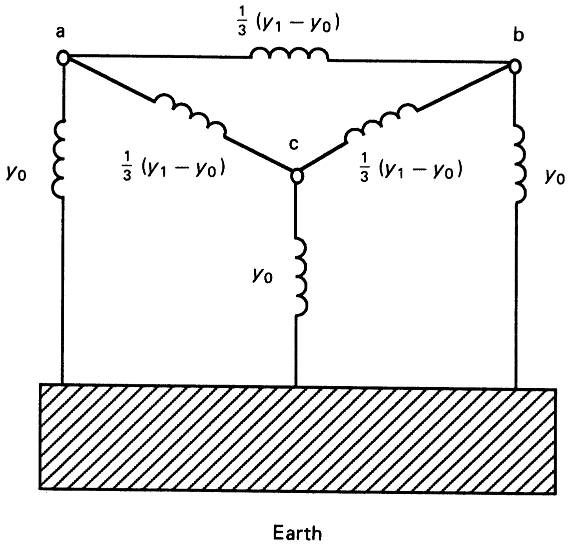


Figure 3.58 Phase representation of core branch impedances

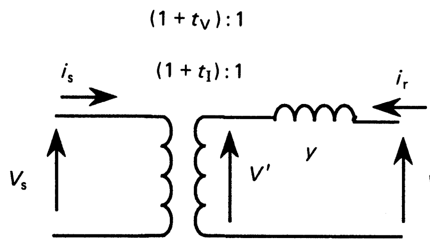


Figure 3.59 Per-unit schematic representation of a single-phase phase-shifting transformer

an ideal phase-shifting transformer. The invariance of the product $V_1{}^*$ across the ideal transformer requires a distinction to be made between the turns ratios for current and voltage; thus

$$V_s=(1+t_v)V'^{\Leftarrow}$$
$$i_r=-(1+t_I)i_s$$

where $1+t_v=1+t+jq$ and $1+t_I=1+t-jq$.

Following the same procedure as before yields the 4×4 phase admittance matrix $\boldsymbol{Y}$ of the equivalent phase-shifting transformer. For a phase-shifting transformer, however, although an equivalent lattice network corresponding to the admittance matrix $\boldsymbol{Y}$ can be constructed, it is no longer a bilinear network because of asymmetry in $\boldsymbol{Y}$. The equivalent circuit of a single-phase phase-shifting transformer is thus of limited value, and the transformer is best represented algebraically by its admittance matrix.

(2) *Three-phase phase-shifting transformers*—For both non-phase-shifting and phase-shifting transformers the phase-admittance matrix for any polyphase bank can be built up from the single-phase admittance matrices by identification of the nodes of each single-phase unit with the three-phase terminations, A, B, C and a, b, c according to the winding connection.

The three-phase admittance matrix of the star–delta phase-shifting transformer is:

$$
\boldsymbol{Y}_{\boldsymbol{3phase}} =
\begin{array}{c}
\\ \text{A} \\ \text{B} \\ \text{C} \\ \text{N} \\ \text{a} \\ \text{b} \\ \text{c}
\end{array}
\begin{array}{c}
\begin{array}{ccccccc} \text{A} & \text{B} & \text{C} & \text{N} & \text{a} & \text{b} & \text{c} \end{array} \\
\begin{pmatrix}
\frac{y}{\alpha_I\alpha_V} & 0 & 0 & \frac{-y}{\alpha_I\alpha_V} & 0 & \frac{y}{\alpha_I\beta} & \frac{-y}{\alpha_I\beta\psi} \\
0 & \frac{y}{\alpha_I\alpha_V} & 0 & \frac{-y}{\alpha_I\alpha_V} & \frac{-y}{\alpha_I\beta} & 0 & \frac{y}{\alpha_I\beta} \\
0 & 0 & \frac{y}{\alpha_I\alpha_V} & \frac{-y}{\alpha_I\alpha_V} & \frac{y}{\alpha_I\beta} & \frac{-y}{\alpha_I\beta\psi} & 0 \\
\frac{-y}{\alpha_I\alpha_V} & \frac{-y}{\alpha_I\alpha_V} & \frac{-y}{\alpha_I\alpha_V} & \frac{3y}{\alpha_I\alpha_V} & 0 & 0 & 0 \\
0 & \frac{-y}{\alpha_V\beta} & \frac{y}{\alpha_V\beta\psi} & 0 & \frac{2y}{\beta^2} & \frac{-y}{\beta^2} & \frac{-y}{\beta^2} \\
\frac{y}{\alpha_V\beta} & 0 & \frac{-y}{\alpha_V\beta\psi} & 0 & \frac{-y}{\beta^2} & \frac{2y}{\beta^2} & \frac{-y}{\beta^2} \\
\frac{-y}{\alpha_V\beta} & \frac{y}{\alpha_V\beta} & 0 & 0 & \frac{-y}{\beta^2} & \frac{-y}{\beta^2} & \frac{2y}{\beta^2}
\end{pmatrix}
\end{array}
$$

The neutral node N has been preserved in the matrix. If N was earthed through an earthing impedance, the appropriate admittance would appear in the element y_{NN} or, if earthed directly, the row and column corresponding to N would be removed.

3.3.5.2 Fault analysis

By representing polyphase network conditions in terms of their phase co-ordinates, i.e. phase voltages, currents and impedances, thereby preserving the physical identity of the system, instead of transforming the phase co-ordinates to symmetrical component co-ordinates, a generalised analysis of polyphase networks under all fault conditions can be developed.

General form of prefault equations It was shown in Section 3.3.3.3 that the general form of the nodal admittance equations $\boldsymbol{YV}=\boldsymbol{I}$ may be used to describe the three-phase system where each bus-bar in the one-line diagram of the balanced system is replaced by three equivalent separate-phase bus-bars. Each voltage and current element in equations $\boldsymbol{YV}=\boldsymbol{I}$ for the balanced system is replaced correspondingly by three phase-to-earth voltages and three currents, with each element of the nodal-admittance matrix being replaced by a three-phase element represented by a 3×3 nodal admittance submatrix. The same principle used in the assembly of the single-phase admittance matrix underlies the assembly of the three-phase admittance matrix.

The phase relationships at each bus-bar are, then, at bus-bar k, for example,

$$I_k = S_k^*/V_k^*$$

where I_k is the phase current injected into bus-bar k, V_k is the phase-to-earth voltage, and S_k is the phase power.

The network admittance matrix $\boldsymbol{Y}$ can be adjusted to the basic prefault form from which all calculations for the various faults commence by classifying the energy sources as active or passive, according to their behaviour during fault conditions. Some loads, for example, may be characterised by passive admittances per phase; thus the equation for node j becomes

$$I_j = y'_{j0}V_j$$

The network nodal admittance equations can be modified accordingly by substituting for the load currents and then transferring the admittances across to supplement the diagonal elements of matrix $\boldsymbol{Y}$. If the load has unequal positive-negative-, and zero-sequence admittances, the admittance y_{j0}' may be replaced by an equivalent 3×3 phase admittance matrix, which in turn is transferred to supplement the appropriate block diagonal 3×3 submatrix of $\boldsymbol{Y}$. In this case, the substitution for the three corresponding phase currents is made simultaneously.

With appropriate node connections, active sources such as machines may be represented by the general network element shown previously containing voltage sources in series with a passive network. The equations $\boldsymbol{YV}=\boldsymbol{I}$ governing the current injected into the network from a star-connected machine connected to the three-phase bus-bars p, q, and r with a neutral earthed through an impedance y_{N0}, are:

$$
\begin{pmatrix}
Y_{pp} & Y_{pq} & Y_{pr} & -y_0 \\
Y_{qp} & Y_{qq} & Y_{qr} & -y_0 \\
Y_{rp} & Y_{rq} & Y_{rr} & -y_0 \\
-y_0 & -y_0 & -y_0 & Y_{NN}
\end{pmatrix}
\begin{pmatrix} V_1 \\ V_2 \\ V_3 \\ V_N \end{pmatrix}
=
\begin{pmatrix}
y_1 & E_p \\
y_1 & E_q \\
y_1 & E_r \\
-y_0 & E_N
\end{pmatrix}
$$

where I_N is the current injected into the neutral node N, and is usually zero, E_N is the sum of the phase e.m.f.s, and is also usually zero ($E_p + E_q + E_r = 0$), y_0 is the machine zero sequence admittance, y_1 is the positive-sequence admittance, and $Y_{NN} = 3y_0 + y_{N0}$. The values of the voltage sources E_p, E_q and E_r are the appropriate phase-displaced transient or subtransient values according to the purpose of the study.

Substituting again for the respective currents in equation $\boldsymbol{YV}=\boldsymbol{I}$ and transferring the product terms $y_{ij}V_j$ to the left-hand side, the modified network admittance equations become

$$\boldsymbol{Y}'\boldsymbol{V} = [\boldsymbol{y}_{\boldsymbol{g}}\boldsymbol{E}] = \boldsymbol{I}' \tag{3.5}$$

where $\boldsymbol{Y}'$ is the supplemented phase admittance matrix and $[\boldsymbol{y}_{\boldsymbol{g}}\boldsymbol{E}]$ or $\boldsymbol{I}'$ is a column matrix the elements of which are of the form y_1E, or zero. The similarity is evident between the three-phase prefault equations and the familiar equivalent one-line diagram or single-phase prefault equations established for three-phase fault studies of balanced systems described in Section 3.3.3.

Solution of equations for various types of fault condition All the various types of fault condition can be analysed by means of simple modifications to, and the solution of, these equations.

(*a*) *Single phase-to-earth fault.* If a single phase-to-earth short circuit occurs at bus-bar k, the bus-bar voltage V_k will be constrained to be zero, the value of the earth reference voltage. To obtain the extra degree of freedom for this constraint to be valid within a consistent set of equations, the current on the right-hand side of the kth equation in equation (3.5) must be unspecified, and allowed to take its value according to the solution of the remaining set of $(n-1)$ equations in the $(n-1)$ unknown voltages, V_j, $i = 1, 2, \ldots, n$, $(i \neq k)$, with $V_k = 0$.

Letting the value of this kth current be I''_k, where

$$I''_k = \Sigma Y'_{km} V_m$$
$$= I'_k + I_{SC}$$

I'_k is the prefault value and Y'_{km} $(m = 1, 2, \ldots, n)$ are the elements of the kth row of the matrix $\boldsymbol{Y}'$, then I_{SC} is the current in the short-circuit connection to earth.

This calculation is exactly the same as that for a three-phase fault in existing computer programs based on a one-line diagram and three-phase apparent-power base.

The numerical methods of solution are the standard methods developed for such problems in linear algebra, namely matrix-inversion methods or, for large systems, iterative methods such as the Gauss–Seidel technique, and can be found in the usual numerical-analysis textbooks.

(*b*) *Multiphase faults.* These are described in (1)–(4) below.

(1) *Phase-to-phase short circuits*—for bus-bars short-circuited together (zero-impedance connection) either the modified admittance equations can be solved subject to a number of additional constraints, or the admittance matrix $\boldsymbol{Y}'$ can be modified, and the number of equations reduced for solution without constraints.

In the first approach, where the number of nodes remains constant, the appropriate bus-bar voltages are constrained to be equal. To obtain the extra degrees of freedom for these constraints to be valid within a consistent set of equations, the currents on the right-hand sides of the corresponding equations must be unspecified and allowed to take their values accordingly. These currents are, at bus-bar k, for example,

$$I''_k = I'_k + \Sigma I_{SC}$$

where ΣI_{SC} is the total short-circuit current injected into bus-bar k from all other bus-bars short circuited to bus-bar k.

For a phase-to-phase short circuit occurring between bus-bars j and k, for example, the equations are solved with voltages $V_j = V_k$ and currents I''_j and I''_k being unknown where

$$I''_j = I'_j + I_{kj}$$

and

$$I''_k = I'_k + I_{jk}$$

I_{kj} $(= -I_{jk})$ is the short-circuit current passing between bus-bars j and k, I_{kj} that being injected into bus-bar j, and I_{jk} being that injected into bus-bar k.

(2) *Phase-to-phase faults via impedances*—if the faults between phases occur through impedances the equations are solved for the voltages $\boldsymbol{V}$ with matrix $\boldsymbol{Y}'$ supplemented by the appropriate admittances.

(3) *Phase-to-phase-to-earth faults*—earthed multiphase faults follow as extensions of the previous sections. If several phases are short circuited to earth simultaneously, the equations are solved with all the respective voltages having a value of zero. The total fault current to earth follows from the appropriate sums of the respective fault currents. If impedances are present in the fault, the corresponding admittances are included in the matrix $\boldsymbol{Y}'$ and the fault currents are evaluated depending on the fault impedance configuration.

(4) *Simultaneous faults*—using phase co-ordinates, any two- or three-phase fault may be considered to be a multiple fault, in the sense that more than one represented bus-bar is involved. The techniques of solution above may be applied, therefore, without restriction, to any number of simultaneous faults, regardless of their type or geographical location. As noted previously, the solution involves the solving of a set of simultaneous linear algebraic equations with or without constraints on the appropriate voltages, depending on the modifications made to the original network connection table.

(*c*) *Open conductors.* Open conductors present no difficulty other than that of introducing an extra bus-bar or bus-bars into the network, depending on the number of open circuits. The appropriate changes are made in the connection table, and the admittance or impedance matrices are modified accordingly.

The source transformation method of solution The above methods of solution may be referred to as 'distributed source methods', in which the various equivalent-current sources retain their identity and are generalisations of the existing methods of solution in the standard three-phase fault-level analysis described in Section 3.3.3.2.

An alternative approach would be to use Norton's theorem, or superposition methods, commencing from the supplemented nodal impedance matrix $\boldsymbol{Z}'$ $(= \boldsymbol{Y}'^{-1})$. The method depends on knowing the voltage drop caused by the fault current, and hence determines this current for an equivalent current source acting alone at the point of fault. The fault conditions are then obtained by superposition.[3]

3.3.5.3 Polyphase loadflow analysis

For polyphase load flow, several questions are raised which are not encountered in balanced one-line diagram analysis. If the phase voltages are unbalanced then the characteristics of the loads under such conditions should be known. An admittance matrix representation may well be better than a specified $P + jQ$ demand. A further problem is the lack of knowledge of the active and reactive power distribution between the phases of the generators.

A synchronous generator model avoiding this latter difficulty and using only the total output power of the machine is, for a machine connected to the three-phase bus-bars p, q and r:

$$\begin{bmatrix} \begin{pmatrix} & & & -y_1 & -y_0 \\ & \boldsymbol{Y}_{abc} & & -\alpha^2 y_1 & -y_0 \\ & & & -\alpha y_1 & -y_0 \\ -y_1 & -\alpha y_1 & -\alpha^2 y_1 & 3y_1 & 0 \\ -y_0 & -y_0 & -y_0 & 0 & (3y_0 + y_{N0}) \end{pmatrix} \begin{pmatrix} V_p \\ V_q \\ V_r \\ E_a \\ V_N \end{pmatrix} \end{bmatrix} = \begin{bmatrix} \begin{pmatrix} I_p \\ I_q \\ I_r \\ \Sigma S^*/E^*_a + y_{N0}|V_N|^2/E^*a \\ 0 \end{pmatrix} \end{bmatrix}$$

To a first approximation the term $y_{N0}|V_N|^2/E^*a$ can be neglected in comparison with the magnitude of the term $\Sigma S^*/E^*_a$.

For each generator the machine admittance matrix $\boldsymbol{Y}_{abc}$ can be added to the polyphase admittance matrix representing the network of lines, transformers and admittance loads. By any of the usual load-flow techniques these equations can then be solved iteratively. At generator bus-bars where a $P|V|$ specification is given, the value of P is the total active power output of the machine and $|V|$ can be either E_a or V_p or any other controlled bus-bar voltage. Several existing conventional (one-line diagram) load-flow programs already allow the generator terminal voltage magnitude to be unknown and to be adjusted according to the voltage value of a remote-controlled bus-bar. In principle, the same situation and adjustments characterise the phase co-ordinate load-flow analysis

3.3.6 Network power limits and stability

Steady-state a.c. power transfer over transmission links has limitations imposed by terminal voltages and link impedance. Transient conditions for stable operation are dynamic, and more complicated.

3.3.6.1 Steady-state conditions

Two typical cases concern a link transferring power from a sending-end generator at bus-bar voltage V_s to a receiving end of voltage V_r where there is either (1) a static load only or (2) a generator. Case (2) is the more important, as loss of synchronism is possible.

(*1*) *Load stability* The power taken by a static load of constant power factor is proportional to the square of the voltage. As the load power is increased the voltage falls, at first slightly but subsequently more rapidly until maximum power is attained. Thereafter both load voltage and power decrease, but the system is still stable, though overloaded. The condition could occur following the clearance of a system fault.

The load is rarely purely static: it usually contains motors. With induction motors the reactive-power requirements increase as the voltage falls, and beyond the maximum-power conditions the machines will stall, and will draw heavy 'pick-up' currents after a restoration of the voltage.

(*2*) *Synchronous stability* The receiving-end active and reactive powers in terms of V_s and V_r, and the parameters ***ABCD***, are given for a transmission link in Section 3.2.13.1. For a short line, $\boldsymbol{A} = 1\angle 0°$⇐and $\boldsymbol{B} = Z\angle\beta$, where $Z = R + jX$ and $\beta = \arctan(X/R)$, conditions shown in *Figure 3.27*.

If the resistance R can be neglected (as is often the case, especially where the link includes terminal transformers), the receiving-end active power P_r and its maximum P_{rm} become

$$P_r = (V_s V_r / X) \sin \theta\psi$$
$$P_{rm} = V_s V_r / X$$

To attain maximum active power, the receiving end must also accept a leading reactive power $Q_r, = V_r^2/X$.

Interpreting the angle $\theta\psi$between V_s and V_r as that between the generator rotor (indicated by the e.m.f. E_t) and V_r, and including the appropriate generator reactance in X, the angle is now the *load angle* δ, and maximum active power transfer will occur for a load angle $\delta = \pi/2$ rad (90°⇐ electric). The relation for normal conditions is marked N in *Figure 3.60*.

Although a system does not operate under continuous steady-state conditions with a system fault, the power–angle relation is important in the assessment of transient stability. The network for which the curve is calculated is obtained by connecting a 'fault shunt' Z_f at the point of fault. The value of Z_f is in terms of Z_- and Z_0, respectively, the total impedance to n.p.s. and z.p.s. currents up to the point of fault. These values are given below for line-line (LL), single-earth (LE), double-earth (LLE) and three-phase (3P) faults, while the corresponding power-angle relations are shown in *Figure 3.60*.

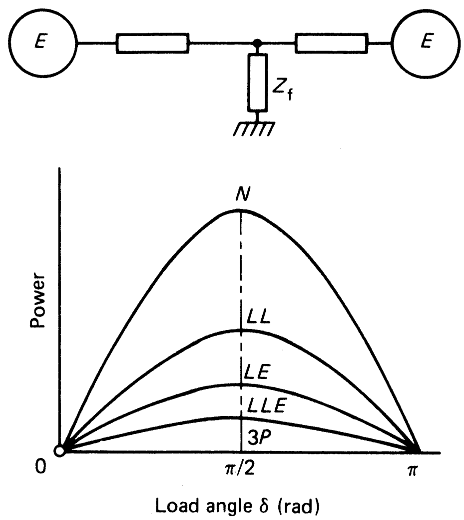


Figure 3.60 Power–angle relationships

Fault:	LL	LE	LLE	3P
Z_f	Z_{-}	$Z_- + Z_0$	$Z_- Z_0/(Z_- + Z_0)$	Zero

3.3.6.2 Transient conditions

If a system in a steady state is subjected to a sudden disturbance (e.g. short circuit, load change, switching out of a loaded circuit) the power demand will not immediately be balanced by change in the prime-mover inputs. To restore balance the rotors of the synchronous machines must move to new relative angular positions; this movement sets up angular oscillations, with consequent oscillations of current and power that may be severe enough to cause loss of synchronism. The phenomenon is termed *transient instability*.

Rotor angle For a single machine connected over a transmission link to an infinite bus-bar, the simple system shown in *Figure 3.61* applies. The mechanical input P_m is, in the steady state, balanced by the electrical output for the angle δ_0 on the full-time power–angle relationship. If an electrical disturbance occurs such that the power–angle relation is suddenly changed to that indicated by the broken curve, the angle cannot

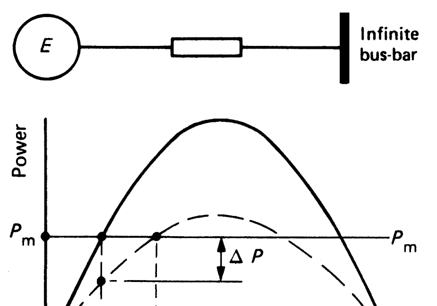


Figure 3.61 A single-machine system

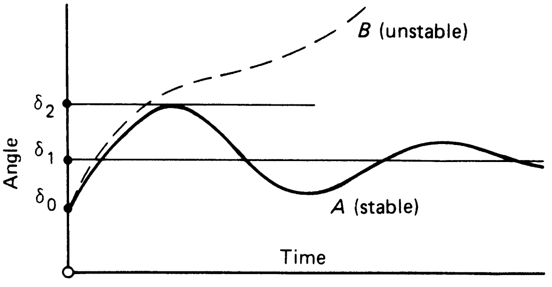


Figure 3.62 Swing curves

immediately change because of the inertia of the generator and prime-mover rotors. The electrical power drops, and a power difference ΔP appears, accelerating the rotating members towards the new balancing angle δ_1. Overshoot takes the rotor angle to δ_2. If the disturbance is not severe, the rotor assumes the angle δ_1 after some rapidly decaying oscillations of frequency 1 or 2 Hz. The angle–time relationship is that shown as curve A in *Figure 3.62*. However, if ΔP is large, the overshoot may cause loss of synchronism—the unstable curve B. A comprehensive investigation of stability thus involves the calculation of *swing* curves for the machines concerned.

Equation of motion The equation of motion for a *single* machine is

$$M(\mathrm{d}^2\delta/\mathrm{d}t^2) + K_1(\mathrm{d}\delta/\mathrm{d}t) + K_2 = \Delta P$$

where M is the angular momentum. If the damping coefficients K_1 and K_2 are ignored, the equation of motion reduces to $\mathrm{d}^2\,\delta/\mathrm{d}t^2 = \Delta\ P/M$.

A mass of inertia J rotating at angular speed ω_1 stores a kinetic energy $W = \frac{1}{2}J\omega^2$. The momentum $M = J\omega_1$ can be usefully related to the machine rating S by the *inertia constant*

$$H = W/S = \tfrac{1}{2}M\omega_1/S = \tfrac{1}{2}J\omega_1^2/S \simeq 20Jn_1^2/S$$

in which ω_1 is the synchronous angular speed (rad/s) and $n_1 = \omega_1/2\pi = f/p$ is the corresponding rotational speed (revolutions per second) for a machine with $2p$ poles operating at a frequency f. The magnitude of H (in joules/volt-ampere, or MJ/MV-A, or seconds) has the typical values given in *Table 3.11*.

A direct solution of the equation of motion is not normally possible, and a *step-by-step* process must be adopted. For this, a succession of time intervals (e.g. 50 ms) is selected and the rotor acceleration ($\mathrm{d}^2\delta/\mathrm{d}t^2$) is calculated at the beginning of each. Assuming the acceleration to be constant throughout a time interval, the angular velocity and the movement $\delta\psi$ during the interval can be found. At the end of the first interval, the new ΔP is obtained from the power–angle curve and used to calculate the acceleration during the second interval, and so on. The complete *swing curve* can thus be obtained. The method can be extended (if the relevant data are available) to include damping, changes in excitation, saliency, prime-mover governor action and other factors that affect the swing phenomenon.

Table 3.11 Inertia constants of 50 Hz synchronous machines

Machine	*n* (rev/s)	*H* (s)
Turbogenerators	50	3–7
	25	5–10
Compensators	—	1–1.25
Motors	—	2–2.25
Hydrogenerators	8.3	2–4
	5	2–3.5
	2.5	2–3
	1.7	15–2.5

Multi-machine system A *two-machine* system with a transmission link can be represented by a single machine feeding an infinite bus-bar and having an equivalent momentum $M = M_1M_2/(M_1 + M_2)$.

A *group* of machines 1, 2, ..., paralleled on the same bus-bar can be treated as a single machine of rating $S = S_1 + S_2 + \ldots$, and of equivalent momentum $M = (S_1/S)M_1 + (S_2/S)M_2 + \ldots$.

For a *multi-machine* network, a separate equation of motion must be set up for each generator and a step-by-step solution undertaken. Determination of ΔP for each machine at the end of a time interval involves a comprehensive load-flow calculation by computer.

Equal-area criterion Neglecting damping, governor action and changes in excitation, the stability of a simple generator/link/infinite-bus-bar system can be checked graphically using power–angle relationships. Consider the system shown in *Figure 3.63*, with the generator operating at a load angle δ_0 on the power–angle curve P_2 with a prime-mover input P_m and both transmission links intact. A fault occurs on one link, changing the power–angle relationship to P_f and giving a power difference ΔP between P_m and the electrical output. As a result the rotor accelerates until the angle δ_s is reached, when the faulted link is switched out. The kinetic energy acquired by the rotor during this period is represented by area A. At δ_s the power–angle relationship becomes P_1 corresponding to a single healthy link. This reverses ΔP and the rotor decelerates. At the angle δ_2 such that area B

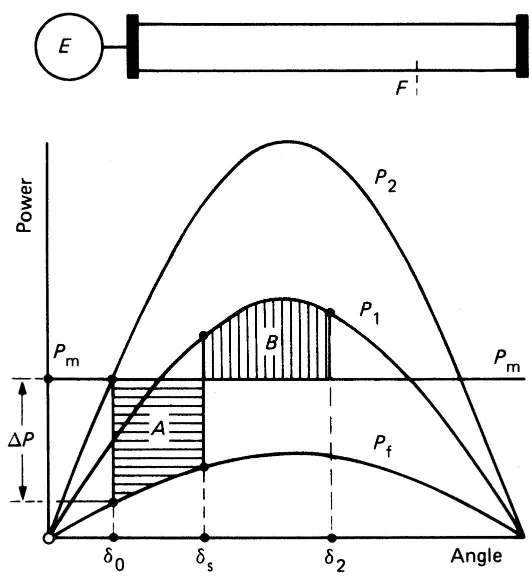


Figure 3.63 Equal-area stability criterion

(representing kinetic energy returned from the rotor) is equal to A, the rotor speed is again synchronous. However, ΔP is now reversed and the rotor will begin to swing back. The range of rotor-angle excursions is stable, but there is a critical value of the fault-clearance angle δ_s that, if exceeded, will result in instability. If all the power–angle relationships are true sinusoids, the critical angle can be found analytically.

Conditions other than that shown in *Figure 3.63* can be dealt with if the relevant power–angle relationships can be drawn. It is to be noted that a *swing* curve may be required to relate rotor angle to *time*, as it is the time of fault clearance (or other event)—a quantity based on the delay of switch opening—that is normally specified.

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